Physics 661 **Homework 1** **Due Wednesday, January 21, 2009**

Problem 1. Two-particle quantum systems with attractive $1/r$ potentials and different masses can be treated by adapting the results from the hydrogen atom, making appropriate changes. For each of the following two-particle systems, find the numerical values of: the effective Bohr radius, the ground-state energy, and the frequency and wavelength of the $n = 2$ to $n = 1$ transitions.

(a) positronium, a bound electron and positron,

(b) deuterium (usually denoted ^{2}H or D, and also called heavy hydrogen), a bound state of a deuteron and an electron,

(c) He^+, a singly ionized helium atom,

(d) a proton and a negatively charged muon,

(e) two neutrons bound together by their gravitational field.

Problem 2. Find $\langle z \rangle$, $\langle z^2 \rangle$, $\langle p_z \rangle$, and $\langle p_z^2 \rangle$ for the ground state of the hydrogen atom. (Write your answers in terms of $\hbar$ and the Bohr radius a_0.) Show that the uncertainty principle is satisfied for Δz and Δp_z.

Problem 3. Expanding the relativistic kinetic energy of a particle of mass m gives:

$$E = \sqrt{(mc^2)^2 + p^2c^2} = mc^2 + \frac{p^2}{2m} - \frac{p^4}{8m^3c^2} + \ldots$$

The first term can be eliminated by shifting the definition of the Hamiltonian by an unobservable constant, since only differences in energy are physically relevant. The second term is the usual non-relativistic kinetic energy. The third term is the leading relativistic correction to the kinetic energy. One can treat its effects by adding a perturbation to the Hamiltonian:

$$H_r = -\frac{(P^2)^2}{8m^3c^2}.$$

Evaluate the corresponding shifts in the energies of the $|n, \ell, m\rangle$ states of the hydrogen atom, using first-order perturbation theory. Hint: to evaluate the matrix elements, it is convenient to write

$$H_r = -\frac{1}{2mc^2}(H_0 + e^2/R)^2$$

where H_0 is the unperturbed Hamiltonian, and then use:

$$H_0|n,\ell,m\rangle = E_n|n,\ell,m\rangle, \qquad \langle n,\ell,m|H_0 = E_n\langle n,\ell,m|,$$

where $E_n = -e^2/(2a_0n^2)$, along with

$$\begin{aligned}\langle n,\ell,m|\frac{1}{R}|n,\ell,m\rangle &= \frac{1}{a_0n^2},\\ \langle n,\ell,m|\frac{1}{R^2}|n,\ell,m\rangle &= \frac{1}{a_0^2n^3(\ell+1/2)}.\end{aligned}$$

(You may assume these as identities, and do not need to prove them. See, for example, eq. A.6.8 on page 455 of Sakurai.) Your answer should have the form

$$\Delta E_r = \frac{\alpha^2}{n^3}\left[\frac{N_1}{n} + \frac{N_2}{\ell+1/2}\right]\left(\frac{e^2}{2a_0}\right)$$

where $\alpha = e^2/(\hbar c) \approx 1/137$, and N_1 and N_2 are constant rational numbers that you will evaluate. How big are the resulting energy shifts, numerically in eV, for the $n=1$ ground state and the $n=2$ first excited states?

Problem 4. Another relativistic correction to the Hamiltonian for a charged particle in an electric field is the Darwin term. In general, the Darwin term for an electron (with charge $-e$) in the presence of a field $\vec{E}$ is:

$$H_D = \frac{e\hbar^2}{8m^2c^2}\vec{\nabla}\cdot\vec{E}.$$

For a point source charge $+e$, like the proton in the hydrogen atom, one has $\vec{E} = -\vec{\nabla}(e/r)$, so $\vec{\nabla}\cdot\vec{E} = -e\nabla^2(1/r) = 4\pi e\delta^{(3)}(\vec{r})$. Therefore, for the hydrogen atom this becomes

$$H_D = \frac{\pi e^2\hbar^2}{2m^2c^2}\delta^{(3)}(\vec{r})$$

in the position space representation. (You can tell this is a relativistic correction because it vanishes in the limit $c\to\infty$. This term is formally of the same order as the Hamiltonian term H_r of the previous problem.) Find the resulting energy shifts for the $|n,\ell,m\rangle$ states. You should get:

$$\Delta E_D = N_3\delta_{\ell,0}\frac{\alpha^2}{n^3}\left(\frac{e^2}{2a_0}\right)$$

where N_3 is a constant rational number that you will evaluate. How big are the resulting energy shifts, numerically in eV, for the $n=1$ ground state and the $n=2$ first excited states?