Physics 661 **Homework 10** **Due Wednesday April 29, 2009**

Problem 1. Consider scattering by a repulsive spherically symmetric δ-function shell potential:

$$V(r) = \frac{\gamma\hbar^2}{2m}\delta(r-R).$$

where γ is a positive constant. The radial part of the $\ell = 0$ partial wave wavefunction can be written as [see eq. (7.6.33) in Sakurai, page 102 in the notes]:

$$A_0(r) = \begin{cases} N j_0(kr) & \text{(for } r \leq R) \\ e^{i\delta_0}[\cos(\delta_0) j_0(kr) - \sin(\delta_0) n_0(kr)] & \text{(for } r \geq R), \end{cases} \tag{1}$$

where N is a normalization factor.

(a) Solve for N by demanding continuity of the wavefunction. You should find that $N = X\sin(kR+\delta_0)/\sin(kR)$, where X is a quantity that doesn't depend on R.

(b) Find an equation that determines the phase shift δ_0 by integrating the Schrodinger equation [in the form of eq. (1) of Problem 2 of Homework Set 9] over a small interval in r from $R-\epsilon$ to $R+\epsilon$. Simplify your answer to the form:

$$\cos(kR+\delta_0) + n_1\left[\frac{\gamma}{k} + \cot(kR)\right]\sin(kR+\delta_0) = 0,$$

where n_1 is an integer that you will find.

(c) Use trigonometric identities on the result you found in part (b) to write the equation in terms of $\cos(\delta_0)$ and $\sin(\delta_0)$. Use this to solve for $\cot(\delta_0)$. You should find:

$$\cot(\delta_0) = n_2\left(\frac{(\gamma/k) + \cot(kR) + \tan(kR)}{(\gamma/k)\tan(kR)}\right)$$

where n_2 is a certain integer that you will find.

(d) Now suppose that the potential barrier is very high ($\gamma/k \gg 1$) but that $\tan(kR)$ is neither particularly large nor small. Show that in the $\gamma/k \to \infty$ limit, you recover the result for the s-wave phase shift for hard-sphere scattering (Sakurai equation 7.6.44).

(e) You are now done with this problem. Just read the following; you do not need to write anything down.

Now suppose the potential barrier is very high ($\gamma/k \gg 1$) but also $\tan(kR)$ is very small (but non-zero). Now the result of part (c) reduces to:

$$\cot(\delta_0) \approx n_2\left(\cot(kR) + \frac{k}{\gamma}\cot^2(kR)\right)$$

Recall that for a resonance that saturates the partial-wave unitarity bound, $\cot(\delta_0)$ should vanish (see for example page 110 of the notes). Therefore, the condition for resonant scattering is

$$\tan(kR) \approx -\frac{k}{\gamma}.$$

For very small k/γ, the solutions will satisfy:

$$kR \approx n\pi - \frac{k}{\gamma},$$

for integer n, so (solving for k, keeping the first order in $1/\gamma$, and calling the result k_r):

$$k_r \approx \frac{n\pi}{R}\left(1 - \frac{1}{\gamma R}\right).$$

Since k is positive by definition in 3-d scattering problems, n must be a positive integer. Therefore, the energies for resonant scattering (saturating the partial-wave unitarity bound) are:

$$E_r = \frac{\hbar^2 k_r^2}{2m} \approx \frac{\hbar^2 n^2 \pi^2}{2mR^2}\left(1 - \frac{2}{\gamma R}\right) \qquad (n = 1, 2, 3, \ldots),$$

again keeping up to the first order in $1/\gamma$. This can be compared with the quasi-bound state energies for very large γ, which have wavefunctions proportional to

$$\psi(\vec{r}) = \frac{\sin(k_b r)}{k_b r}$$

where $k_b = n\pi/R$ in order to satisfy the boundary condition that the wavefunction vanishes at the infinite barrier at $r = R$. The corresponding energy eigenvalues in that extreme limit are

$$E_b \approx \frac{\hbar^2 k_b^2}{2m} = \frac{\hbar^2 n^2 \pi^2}{2mR^2} \qquad (n = 1, 2, 3, \ldots)$$

This shows that the resonant scattering energies agree with the quasi-bound state energies, at least in the large γ limit. (In the former, we have kept first order in $1/\gamma$ to show off, but not in the latter.)

This problem is a very crude model for neutron-nucleus scattering processes in the real world, which typically feature cross sections with sharp resonances superimposed on functions that are otherwise smoothly falling with energy.

Problem 2. Consider **inelastic** scattering of an electron on a Hydrogen atom in the ground state (1s), leaving the atom in the (2s) state as the final state. (Ignore the fact that the electrons in the atom are identical to the electrons in the beam.)

(a) Show that the differential cross section (in the Born approximation of section 7.12 of Sakurai) for $e^- + (1s) \to e^- + (2s)$ is:

$$\frac{d\sigma}{d\Omega} = N_1 \left(\frac{k'}{k}\right) \frac{a_0^2}{(q^2 a_0^2 + 9/4)^{N_2}}$$

where N_1 and N_2 are integers that you will find, and

$$q^2 = k^2 + k'^2 - 2kk' \cos\theta, \quad \text{with} \quad k'^2 = k^2 - \frac{3}{4a_0^2}.$$

Hint: when integrating to find the form factor, temporarily choose $\vec{q}$ to be in the z direction, since the result depends only on the magnitude of $\vec{q}$. Do the angular integrals first. You may find two of the following integrals useful:

$$\begin{aligned}
\int_0^\infty e^{-Ar} \sin(Br)\, dr &= \frac{B}{A^2 + B^2}, \\
\int_0^\infty e^{-Ar} \sin(Br)\, r\, dr &= \frac{2AB}{(A^2 + B^2)^2}, \\
\int_0^\infty e^{-Ar} \sin(Br)\, r^2\, dr &= \frac{6A^2B - 2B^3}{(A^2 + B^2)^3}.
\end{aligned}$$

(b) You are actually done with this problem. Just read the following facts related to this problem, which are presented for your enjoyment and edification. You do not have to write anything down.

The Born approximation for the differential cross-section found in part (a) gets better in the high-energy limit ($ka_0 \approx k'a_0 \gg 1$), for which $q \approx 2k\sin(\theta/2)$. The differential cross section is dominated by very small angles $\theta \lesssim 1/(ka_0)$, and for larger angles fall off much more rapidly than the elastic scattering case that you studied in Problem 2 on Homework Set 8. For the total inelastic cross section to the (2s) state, carefully integrating the result of part (a) above yields:

$$\sigma(e^- + (1s) \to e^- + (2s)) = \left(\frac{2}{3}\right)^{10} \frac{128\pi}{5k^2} \qquad (\text{for } ka \gg 1).$$

This can be compared with the result for elastic scattering in the high-energy limit, which from part (e) of Problem 2 on Homework Set 8 can be found to be:

$$\sigma(e^- + (1s) \to e^- + (1s)) = \frac{7\pi}{3k^2} \qquad (\text{for } ka \gg 1).$$

Numerically, the elastic total cross section is larger by a factor of roughly 5.

Problem 3. Consider a spinless particle of mass m moving in one dimension in an infinite-height square well potential of length L:

$$V(x) = \begin{cases} \infty & (\text{for } x < 0) \\ 0 & (\text{for } 0 < x < L) \\ \infty & (\text{for } x > L) \end{cases}$$

(a) What are the properly normalized energy eigenstate wavefunctions for this potential?

(b) Use equation (2.5.8) in Sakurai to find the propagator $K(x'', t''; x', t')$ for the particle moving in this potential. Your answer should involve an infinite sum $\sum_{n=1}^{\infty}$.

(c) Show that the specific result you got for part (b) obeys the composition law for propagators:

$$K(x'', t''; x', t') = \int_{-\infty}^{\infty} dx\, K(x'', t''; x, t) K(x, t; x', t')$$

for any fixed t. [Hint: use $\int_0^L dx \sin(n\pi x/L) \sin(n'\pi x/L) = \delta_{n,n'} L/2$.]