Physics 661 **Homework 5** **Due Wednesday, February 25, 2009**

Problem 1. A particle of mass m moving in one dimension is in a potential $V(x) = -A\delta(x-a) - A\delta(x+a)$, where $A > 0$.

(a) What is the form of the wave function for a bound state with even parity? Choose your answer so that $\psi(x) = e^{-\kappa x}$ for large positive x. (You do not need to worry about the overall normalization of your wavefunction.)

(b) Find an equation that determines the bound state energies for even parity states, and determine graphically how many even parity bound states there are. [Hint: you should be able to write your equation in the form $\kappa a =$ (polynomial in $e^{-\kappa a}$).]

(c) Repeat parts (a) and (b) for odd parity. For what values of A are there no odd-parity bound states?

(d) Solve for the even parity bound state energy analytically in the limit of $mAa/\hbar^2 \ll 1$.

(e) Find the even and odd parity state binding energies in the limit $mAa/\hbar^2 \gg 1$. What happens to the energy splitting as $a \to \infty$?

Problem 2. Consider a particle of mass m moving in one dimension in the presence of a periodic potential $V(x) = V(x+a)$. The Bloch wavefunction is $\psi(x) = u(x)e^{ikx}$, where $u(x)$ is a periodic function with $u(x) = u(x+a)$.

(a) Starting from the Schrodinger equation, find the second-order differential equation that governs the functions $u(x)$, for a given k and energy E.

(b) The periodic potential $V(x)$ can be expanded in a Fourier series as follows $V(x) = \sum_{n=-\infty}^{\infty} V_n \exp(2\pi inx/a)$. Expand the periodic coefficient $u(x)$ in a similar series, $u(x) = \sum_{n=-\infty}^{\infty} u_n \exp(2\pi inx/a)$, and convert the differential equation you found in part (a) into an infinite set of coupled equations for the coefficients u_n, of the form

$$c_n u_n = \sum_{\ell=-\infty}^{\infty} V_{n-\ell} u_\ell \qquad \text{(for each } n\text{)},$$

where the coefficients c_n are quantities that you will determine.

Problem 3. Consider the 1-d Kronig-Penney δ-function model studied in class, for which we found $E = \hbar^2 q^2/2m$ with the dispersion relation

$$\cos(ka) = \cos(qa) + \left(\frac{mv_0 a}{\hbar^2}\right)\frac{\sin(qa)}{qa}.$$

We noted the existence of a series of allowed energy bands, which one can write as follows:

$$\begin{aligned} E_1^{\text{low}} &< E < E_1^{\text{high}} && \text{(band 1)} \\ E_2^{\text{low}} &< E < E_2^{\text{high}} && \text{(band 2)} \\ E_3^{\text{low}} &< E < E_3^{\text{high}} && \text{(band 3)} \end{aligned}$$

etc.

(a) Even though the dispersion relation is a transcendental equation for q, you can find very simple analytic expressions for the E_n^{high}. Do so. [Hint: what happens when $\sin(qa) = 0$?]

(b) Using the expressions for E_n^{high}, show that

$$E_{n+1}^{\text{low}} = \frac{\hbar^2}{2m}\left(\frac{n\pi}{a} + \Delta q_n\right)^2,$$

where Δq_n is the smallest positive solution to the equation

$$\cos(\Delta q_n) + (\text{stuff})\sin(\Delta q_n) = N,$$

where "stuff" is an expression (dependent on n and Δq_n) that you will determine, and N is a certain constant integer.

(c) In the limit of small v_0, find a simple approximate expression for the lowest allowed energy E_1^{low}.