

Problem 1. The Lippmann-Schwinger scattering equation formalism can also be applied to a one-dimensional quantum mechanical scattering from a potential  $V(x)$  that vanishes outside of some range  $-a < x < a$ . In this problem we will treat this in the position representation (using a slightly different method than used for the 3-d problem).

(a) Suppose we have an incident wavefunction coming from the left of the form  $e^{ikx}$ . (The normalization here is arbitrary, and won't matter for this problem.) Show that if the full wavefunction obeys

$$\psi(x) = e^{ikx} + \frac{2m}{\hbar^2} \int_{-a}^a dx' G(x, x') V(x') \psi(x'), \quad (1)$$

then it satisfies the time-independent Schrodinger equation

$$-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \psi(x) = [E - V(x)] \psi(x), \quad (2)$$

provided that  $E = \hbar^2 k^2 / 2m$  and the Green function satisfies the differential equation

$$\left( \frac{\partial^2}{\partial x^2} + k^2 \right) G(x, x') = \delta(x - x'). \quad (3)$$

(b) Solve eq. (3) for the Green function by trying a solution of the form:

$$G(x, x') = \begin{cases} A e^{ik(x-x')} & (\text{for } x \geq x') \\ A e^{-ik(x-x')} & (\text{for } x \leq x') \end{cases} \quad (4)$$

and solving for the constant  $A$ . [Hint: integrate with respect to  $x$  over an infinitesimal range that includes the point  $x = x'$ .]

(c) Consider the special case of an attractive  $\delta$  function potential:

$$V(x) = -\frac{\gamma \hbar^2}{2m} \delta(x), \quad (5)$$

where  $\gamma$  is a positive constant. Solve the integral equation (1) for  $\psi(x)$ . [Hint: after integrating over the delta function, first solve the resulting equation for  $\psi(0)$ .]

(d) Check your answer for part (c) by instead directly solving the time-independent Schrodinger equation for  $\psi(x)$ , using a guess of the form:

$$\psi(x) = \begin{cases} e^{ikx} + C e^{-ikx} & (\text{for } x < 0) \\ D e^{ikx} & (\text{for } x > 0) \end{cases} \quad (6)$$

and solving for the constants  $C$  and  $D$ . Write  $C$  and  $D$  in a form with real denominators. Find the reflection and transmission probabilities  $R = |C|^2$  and  $T = |D|^2$ , and show that

$R + T = 1$ . They can be interpreted as the probabilities for an incoming particle to be reflected or transmitted from the potential.

(e) The potential of eq. (5) has exactly one bound state wavefunction solution. Find its wavefunction and energy, and write the latter as  $E = -\frac{\hbar^2 \kappa^2}{2m}$  where  $\kappa$  is a quantity that you will determine.

[Note that both  $R$  and  $T$ , when viewed as functions of  $k$  taken to be a complex variable, have poles (divergences) for  $k = i\kappa$ . This is an example of a more general theme that we will discuss in class in the case of 3-d scattering: the scattering amplitudes for positive energy wavefunctions, when analytically continued to negative energies, have poles at the bound state energies.]

(f) Now let's go back to the case of a general potential. Use the results you found for parts (a) and (b) [and use the special case of parts (c) and (d) as a guide] to find the reflection probability  $R$  in the Born approximation where  $\psi(x') \approx e^{ikx'}$  in equation (1), to second order in the potential  $V$ . You should obtain the result:

$$R = X \left| \int_{-a}^a dx' e^{2ikx'} V(x') \right|^2, \quad (7)$$

where  $X$  is a quantity that you will find. [If you try to find  $T$  to the same order, you may find it somewhat trickier, unless you use the easy method  $T = 1 - R$ . That's because finding  $T$  directly to second order in  $V$  requires going beyond leading order in the Born approximation. You don't have to do this, but you might want to think about why.]

**Problem 2.** Consider the scattering of particles of mass  $m$  and incident energy  $E$  on a spherical potential

$$V(r) = \begin{cases} V_0 & (\text{for } r < a) \\ 0 & (\text{for } r > a) \end{cases}$$

Here  $V_0$  is a constant (negative for an attractive potential and positive for a repulsive one) and  $a$  is the radial size of the potential. In the following, use the Born approximation.

(a) Find the differential scattering cross-section for small  $|V_0|$ . (Check the units!)

(b) Show that in the limit of small  $ka$  (where  $E = \hbar^2 k^2 / 2m$ ), the differential cross-section you found in part (a) is constant with respect to the scattering angle, and evaluate the total cross section. You should find  $\sigma = AV_0^2 a^6$  where  $A$  is a quantity that you will evaluate. (Check the units!)

(c) Working now to the next-to-leading order in an expansion in small  $ka$ , show that the differential cross-section has the form  $\frac{d\sigma}{d\Omega} = B + C \cos(\theta)$ , and determine the quantities  $B$  and  $C$ . (Check the units!)