

Problem 1. A graduate student is assigned to measure the differential cross-section of a beam of low-energy particles with known kinetic energy E on an unknown target of spherically symmetric atoms. She finds that the differential cross-section is not isotropic, and can be written as:

$$\frac{d\sigma}{d(\cos\theta)} = \frac{\pi\hbar^2}{mE} [c_0 + c_1 \cos(\theta) + c_2 \cos^2(\theta)].$$

where c_0 , c_1 , and c_2 are measured quantities that depend on E but not on θ . Assume that the target is very small compared to the experimental apparatus, and that multiple scatterings within the target are not an issue. Also assume that the results should be interpreted as a combination of S -wave and P -wave scattering.

- What are the dimensions of c_0 , c_1 , and c_2 ?
- Find real equations relating the given quantities to the phase shifts δ_0 and δ_1 .
- Use partial wave unitarity to find theoretical upper bounds on c_0 and c_2 .
- What are the theoretical upper and lower bounds on c_1 ? [Hint: both bounds are realized, for example, by phase shifts satisfying $\delta_0 = \pi n_0/6$ and $\delta_1 = \pi n_1/6$ where n_0 and n_1 are certain positive integers less than 6. You may use this fact without proving it.]

Problem 2. Recall that in the case of scattering of particles of energy E from a spherically symmetric potential $V(r)$, we write the partial wave expansion of the scattering wavefunction as:

$$\psi^{(+)}(\vec{r}) = \frac{1}{(2\pi)^{3/2}} \sum_{\ell=0}^{\infty} i^\ell (2\ell + 1) A_\ell(r) P_\ell(\cos\theta).$$

[See eq. (7.6.29) on page 405 of Sakurai.] Using Schrodinger's equation, it follows that each partial-wave radial wavefunction $A_\ell(r)$ satisfies:

$$\frac{d}{dr} \left(r^2 \frac{dA_\ell}{dr} \right) = \left[\frac{2m}{\hbar^2} V(r) + \frac{\ell(\ell + 1)}{r^2} - k^2 \right] r^2 A_\ell, \quad (1)$$

where $E = \hbar^2 k^2 / 2m$.

- Do a change of variables $x \equiv kr$ to eliminate r , and find the resulting differential equation in x for A_ℓ , in terms of $V(x/k)$.
- For $V(r) = 0$, show by direct substitution that the equation you got in part (a) has solutions

$$j_0(x) = \frac{\sin(x)}{x} \quad \text{and} \quad n_0(x) = -\frac{\cos(x)}{x} \quad (\text{for } \ell = 0),$$

$$j_1(x) = \frac{\sin(x)}{x^2} - \frac{\cos(x)}{x} \quad \text{and} \quad n_1(x) = -\frac{\cos(x)}{x^2} - \frac{\sin(x)}{x} \quad (\text{for } \ell = 1).$$

(You don't have to do it, but more generally you can prove by substitution that for any given ℓ , the functions

$$j_\ell(x) = (-x)^\ell \left(\frac{1}{x} \frac{d}{dx} \right)^\ell \left(\frac{\sin(x)}{x} \right),$$

$$n_\ell(x) = -(-x)^\ell \left(\frac{1}{x} \frac{d}{dx} \right)^\ell \left(\frac{\cos(x)}{x} \right)$$

satisfy the equation. These are the spherical Bessel and Neumann functions; in class we asserted that they were the solutions, but we didn't prove it!

Problem 3. Consider the problem of low-energy scattering from a repulsive potential:

$$V(r) = \begin{cases} V_0 & (\text{for } r < R) \\ 0 & (\text{for } r > R) \end{cases}$$

where V_0 is positive and $E < V_0$. This is very similar to a problem we did in class, except that now the solutions inside the potential are exponential rather than oscillatory. Consider only the S -wave contribution to the scattering.

(a) Show that for $r < R$, a solution is $A_0 = \sinh(\kappa' r)/\kappa' r$, up to a multiplicative constant independent of r , by plugging it into eq. (1) of Problem 2. In the process, find κ' in terms of V_0 and E . [There is also a $\cosh(\kappa' r)/\kappa' r$ solution, but it is not used because it is singular at $r = 0$.]

(b) Compute

$$\beta_0 = \left. \frac{r}{A_0} \frac{dA_0}{dr} \right|_{r=R}$$

in terms of κ' and R , and simplify it as much as possible.

(c) Find $\tan(\delta_0)$ in terms of κ' , k , and R . Your expression should only depend on R through \sin , \cos , \sinh , and \cosh functions, and should be simplified as much as possible.

(d) Compute the scattering length a in terms of κ' and R , by considering the low-energy limit. Simplify it as much as possible.

(e) What is the total S -wave cross-section in the low-energy limit?

Problem 4. In class it was mentioned that in low-energy neutron-proton scattering with anti-parallel spins, experiment reveals that the measured S -wave cross-section exhibits a "virtual bound state" with energy $E_v = -0.1$ MeV. What is the corresponding scattering length? What is the resulting total np scattering cross-section $\sigma_{S=0}$ for the S -wave with anti-parallel spins in the limit of $E \rightarrow 0$? What is $\sigma_{S=0}$ for $E = 0.5$ MeV? Express your answers as actual numbers with appropriate units.