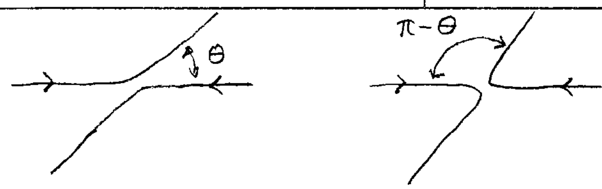



are indistinguishable.

At $\theta = \pi/2$, there is <u>constructive</u> interference: $\left.\frac{d\sigma}{d\Omega}\right|_{90^\circ} = 4|f(\pi/2)|^2$

Also, only <u>even $l$</u> partial waves contribute:

$$f(\theta) + f(\pi-\theta) = \sum_{l=0}^{\infty} i^l (2l+1) P_l(\cos\theta) f_l + \sum_{l=0}^{\infty} i^l (2l+1) \underbrace{P_l(\cos(\pi-\theta))}_{(-1)^l P_l(\cos\theta)} f_l$$

$$= 2 \sum_{l\ \text{even}} i^l (2l+1) P_l(\cos\theta) f_l .$$

Now consider identical spin-½ fermions. There are two cases:

1) If total spin $S=0$, the spin state is antisymmetric. The total wavefunction must be antisymmetric for fermions, so the spatial wavefunction is symmetric. So $\psi(\vec r) = \psi(-\vec r)$, and $\left.\frac{d\sigma}{d\Omega}\right|_{S=0} = |f(\theta) + f(\pi-\theta)|^2$, just like for bosons.

2) If total spin $S=1$, spin state is symmetric $\Rightarrow$ antisymmetric spatial wavefunction $\Rightarrow$ $\psi(\vec r) = -\psi(-\vec r)$ (odd parity).

Therefore $\psi(\vec r) = e^{i\vec k\cdot\vec r} - e^{-i\vec k\cdot\vec r} + [f(\theta) - f(\pi-\theta)]\frac{e^{ikr}}{r}$

and $\left.\frac{d\sigma}{d\Omega}\right|_{S=1} = |f(\theta) - f(\pi-\theta)|^2 = |f(\theta)|^2 + |f(\pi-\theta)|^2 - 2\mathrm{Re}[f^*(\theta) f(\pi-\theta)]$

It follows that $\left.\frac{d\sigma}{d\Omega}\right|_{S=1} = 0$ for $\theta = \frac{\pi}{2}$ ($90^\circ$ scattering)

Also, only <u>$l$=odd</u> partial waves contribute:

$$f(\theta) - f(\pi-\theta) = 2 \sum_{l\ \text{odd}} i^l (2l+1) P_l(\cos\theta) f_l .$$

More typically, the spins are random and unmeasured. The 4 spin states (1 for $S=0$ and 3 for $S=1$) are equally likely, so the observed cross section is the average:

$$\left.\frac{d\sigma}{d\Omega}\right|_{\text{unpolarized}} = \frac{1}{4}\left.\frac{d\sigma}{d\Omega}\right|_{S=0} + \frac{3}{4}\left.\frac{d\sigma}{d\Omega}\right|_{S=1}$$

$$= |f(\theta)|^2 + |f(\pi-\theta)|^2 - \mathrm{Re}\left[f^*(\theta)\, f(\pi-\theta)\right]$$

note no 2!



<u>Inelastic Scattering</u>

So far we've assumed that the final state is the same as the initial, except with redirected momentum. This is elastic scattering.

In high-energy physics experiments, elastic scattering is boring. We care more about things like:

$p\bar{p} \to t\bar{t}$ (Tevatron) or $pp \to H \to \gamma\gamma$ (LHC Higgs discovery mode).

At lower energies,

$e^-$ + atom (ground state) $\to$ $e^-$ + atom (ground state)

is elastic, but we might also care about

$\boxed{e^- + \text{atom (ground state)} \to e^- + \text{atom (excited state)}}$

which is inelastic.

Initial state ket $= |\vec{k}, 0\rangle$

for electron; ground state of atom

Final state ket $= |\vec{k}', n\rangle$ state of atom (elastic if $n \neq 0$).

For normalized wavefunctions, put everything in a large cubic box of side $L$, with periodic boundary conditions for the plane wave electrons.

The corresponding wavefunctions are

$$\frac{1}{L^{3/2}} e^{i\vec{k}\cdot\vec{r}} \psi_0(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_Z) \qquad \text{for initial state}$$

wavevector, position of incident electron

coordinates of target atom electrons

and $\frac{1}{L^{3/2}} e^{i\vec{k}'\cdot\vec{r}} \psi_n(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_Z)$ for final state.

To compute $\frac{d\sigma}{d\Omega}$, use time-dependent perturbation theory.

Find rate $w$ of transition from $|\vec{k}, 0\rangle$ to $|\vec{k}', n\rangle$.

Details in section 7.11 of Sakurai (we'll skip them...)

$$\frac{d\sigma}{d\Omega}(0 \to n) = \frac{k'}{k} L^6 \left| \frac{1}{4\pi} \frac{2m_e}{\hbar^2} \langle \vec{k}', n | V | \vec{k}, 0 \rangle \right|^2$$

Note * this is proportional to matrix element squared.

* $\propto V^2$, so Born approximation.
* $k' \neq k$ here.
* $L^6$ factor is cancelled by $1/L^3$ hidden in $\langle \vec{k}', n | V | \vec{k}, 0\rangle$.
  This is because $\langle \vec{r} | \vec{k} \rangle = \frac{1}{L^{3/2}} e^{i\vec{k}\cdot\vec{r}}$.

For an atom, $V(\vec{r}) = \underbrace{-\frac{Ze^2}{r}}_{\text{nucleus}} + \sum_{i=1}^{Z} \frac{e^2}{|\vec{r} - \vec{r}_i|}$ (if not neutral, sum may not go to $Z$).

position of ith target electron.

Neglect the issue of target electrons and initial electrons being identical. This is cheating, but becomes a better approximation for large $k$ (wavefunction overlap with target electrons is small in the high momentum limit).

So $$\langle \vec{k}', n | V | \vec{k}, 0 \rangle = \int d^3\vec{r} \, \frac{e^{-i\vec{k}'\cdot\vec{r}}}{L^{3/2}} \frac{e^{i\vec{k}\cdot\vec{r}}}{L^{3/2}} \langle n | \left( -\frac{Ze^2}{r} + \sum_i \frac{e^2}{|\vec{r} - \vec{r}_i|} \right) | 0 \rangle$$

$$= \frac{1}{L^3} \int d^3\vec{r} \, e^{i\vec{q}\cdot\vec{r}} \prod_{i=1}^{Z} \int d^3\vec{r}_i \, \psi_n^*(\vec{r}_1, \ldots, \vec{r}_Z) \left[ -\frac{Ze^2}{r} + \sum_i \frac{e^2}{|\vec{r} - \vec{r}_i|} \right] \psi_0(\vec{r}_1, \ldots, \vec{r}_Z)$$

where $\vec{q} = \vec{k} - \vec{k}'$.

First consider the nucleus contribution. The integration separates:

$$\rightarrow \frac{1}{L^3} \underbrace{\int d^3\vec{r}\; e^{i\vec{q}\cdot\vec{r}} \left(-\frac{Ze^2}{r}\right)}_{= -Ze^2 \lim_{\mu\to 0} \underbrace{\int d^3\vec{r}\; \frac{e^{-\mu r + i\vec{q}\cdot\vec{r}}}{r}}_{4\pi/q^2}} \underbrace{\prod_{i=1}^{Z} \int d^3\vec{r}_i\; \psi_n^* \psi_0}_{= \delta_{n,0}}$$

as for Yukawa potential (Sakurai p. 387, these notes p. 90)

So $\rightarrow -\delta_{n,0}\, Ze^2 \left(\frac{4\pi}{q^2}\right)$ contributes to <u>elastic</u> scattering.

Now consider the atomic electron potential contribution:

$$= \frac{e^2}{L^3} \prod_{i=1}^{Z} \int d^3\vec{r}_i\; \psi_n^* \psi_0 \underbrace{\sum_{i=1}^{Z} \int d^3\vec{r}\; \frac{e^{i\vec{q}\cdot\vec{r}}}{|\vec{r}-\vec{r}_i|}}$$

$$= \sum_{i=1}^{Z} \int d^3\vec{r}\; \frac{e^{i\vec{q}\cdot(\vec{r}+\vec{r}_i)}}{|\vec{r}|} = \sum_{i=1}^{Z} e^{i\vec{q}\cdot\vec{r}_i} \underbrace{\int d^3\vec{r}\; \frac{e^{i\vec{q}\cdot\vec{r}}}{r}}_{4\pi/q^2}$$

Putting things together:

$$\langle \vec{k}', n | V | \vec{k}, 0\rangle = \frac{1}{L^3}\left(\frac{4\pi e^2}{q^2}\right)\Bigg[-Z\delta_{n,0} + \underbrace{\prod_{i=1}^{Z}\int d^3\vec{r}_i\; \psi_n^* \psi_0 \left(\sum_{i=1}^{Z} e^{i\vec{q}\cdot\vec{r}_i}\right)}_{= \langle n | \sum_{i=1}^{Z} e^{i\vec{q}\cdot\vec{r}_i} | 0\rangle}\Bigg].$$

So define the <u>form factor</u> for the transition $|0\rangle$ to $|n\rangle$:

$$F_n(\vec{q}) = \frac{1}{Z} \langle n | \sum_{i=1}^{Z} e^{i\vec{q}\cdot\vec{r}_i} | 0\rangle .$$

Then we get $\langle \vec{k}', n | V | \vec{k}, 0\rangle = \frac{1}{L^3}\left(\frac{4\pi Z e^2}{q^2}\right)\left[-\delta_{n,0} + F_n(q)\right].$

Now we can write the differential cross section:

$$\frac{d\sigma}{d\Omega}(0\to n) = \frac{k'}{k}\left|\frac{1}{4\pi}\left(\frac{2m_e}{\hbar^2}\right)\left(\frac{4\pi Z e^2}{q^2}\right)\left[-\delta_{n,0} + F_n(\vec{q})\right]\right|^2$$

$$= \frac{4 m_e^2}{\hbar^4}\frac{(Ze^2)^2}{q^4}\left(\frac{k'}{k}\right)\left|-\delta_{n,0} + F_n(q^2)\right|^2$$

If we only care about inelastic scattering, then we can ignore the $\delta_{n,0}$ part.

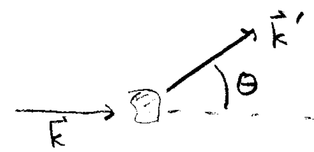
Since $\frac{d\sigma}{d\Omega}$ has units of area, rewrite in terms of $a_0^2$:

$$\frac{d\sigma}{d\Omega}(0\to n) = 4Z^2a_0^2\left(\frac{k'}{k}\right)\frac{1}{(qa_0)^4}|F_n(q)|^2$$

Measuring $\frac{d\sigma}{d\Omega}$ gives information about $F_n(q)$, and so learn about the structure of the target.

Angular dependence comes through

$$q^2 = |\vec{k}-\vec{k}'|^2 = k^2+k'^2-2kk'\cos\theta$$




So $d(\cos\theta) = -\frac{q\,dq}{kk'}$.

Since $\frac{d\sigma}{d\Omega}(0\to n)$ is sharply peaked at small $q$ $\left(\propto \frac{1}{q^4}\right)$, in most collisions the momentum direction doesn't change much.

For very high $q$, the structure of the nucleus is also probed.

Requires $q \gtrsim \frac{1}{10R_{\text{nucleus}}} \sim \frac{1}{10^{-12}\text{cm}} = 10^{12}\ \text{cm}^{-1}$

Then $-\frac{Ze^2}{r} \to -Ze^2\int d^3\vec{r}\ \frac{N(\vec{r}')}{|\vec{r}-\vec{r}'|}$


where $N(\vec{r}')$ = nuclear charge density (units $= \frac{1}{(\text{length})^3}$.)

Require $\int d^3\vec{r}'\ N(\vec{r}') = 1$.

The previous approximation was $N(\vec{r}') = \delta^{(3)}(\vec{r}')$.

So we need to make the replacement:

$$\int d^3\vec{r}\ \ Ze^2\frac{e^{i\vec{q}\cdot\vec{r}}}{r} \to \int d^3\vec{r}\ Ze^2\int d^3\vec{r}'\ \frac{e^{i\vec{q}\cdot\vec{r}}N(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

$$= Ze^2\int d^3\vec{r}\int d^3\vec{r}'\ \frac{e^{i\vec{q}\cdot(\vec{r}+\vec{r}')}N(\vec{r}')}{|\vec{r}|} \qquad \text{(by shifting } \vec{r}\to\vec{r}+\vec{r}'\text{)}.$$

$$= Ze^2\underbrace{\int d^3\vec{r}'\,e^{i\vec{q}\cdot\vec{r}'}N(\vec{r}')}_{\equiv F_{\text{nucleus}}(\vec{q})}\underbrace{\int d^3\vec{r}\ \frac{e^{i\vec{q}\cdot\vec{r}}}{r}}_{=\frac{4\pi}{q^2}}$$

The factor $F_{nucleus}(\vec{q})$ = form factor for the nucleus.

When this is important, $q$ is so large that atomic electrons aren't.

Then $$\frac{d\sigma}{d\Omega} = \underbrace{\left(\frac{d\sigma}{d\Omega}\right)_{Rutherford}}_{\text{due to a point-like nucleus with charge } +Ze.} |F_{nucleus}(\vec{q})|^2$$

If $q$ isn't too large (say, $q R_{nucleus} < 0.5$), and $N(\vec{r}) \overset{\text{spherically symmetric}}{=} N(r)$

$$F_{nucleus}(\vec{q}) = \int d^3\vec{r}\, e^{i\vec{q}\cdot\vec{r}} N(\vec{r}) = \int d^3\vec{r}\left(1 + \underbrace{i\vec{q}\cdot\vec{r}}_{\substack{\to 0 \\ \text{after integrating}}} - \frac{1}{2}q^2 r^2 (\hat{q}\cdot\hat{r})^2 + \dots\right) N(r)$$

$$= \underbrace{\int d^3\vec{r}\, N(r)}_{=1} - \frac{1}{2}q^2 \underbrace{\int r^2 dr\, N(r)\, r^2}_{\equiv \frac{1}{4\pi}\langle r^2\rangle_{nucleus}} \underbrace{\int d\phi}_{2\pi} \underbrace{\int d(\cos\theta)\, \cos^2\theta}_{2/3} + \dots$$

$$= 1 - \frac{1}{2}q^2 \langle r^2\rangle_{nucleus} + \dots$$

This has actually been used to measure the "size" of the nucleus (as defined by $\langle r^2\rangle_{nucleus}$).

The very high-energy version of this (using beams of electrons, muons, neutrinos, and their antiparticles) is called "deep inelastic scattering" and has been used to find out the structure of the proton and other nuclei. This is one way we know that the proton contains point-like constituents (quarks).

<u>Propagators</u> Consider a state ket $|\Psi\rangle$ with wavefunction at time $t'$ given by $\Psi(\vec{r}', t')$. What is the wavefunction at a later time $t''$?

$$\Psi(\vec{r}'', t'') = \langle \vec{r}'', t'' | \Psi\rangle = \int d^3\vec{r}' \, \langle \vec{r}'', t'' | \vec{r}', t'\rangle \underbrace{\langle \vec{r}', t' | \Psi\rangle}_{\Psi(\vec{r}', t')}$$

So, define $K(\vec{r}'', t''; \vec{r}', t') = \begin{cases} \langle \vec{r}'', t'' | \vec{r}', t'\rangle & (\text{if } t'' > t') \\ 0 & (\text{if } t'' < t') \end{cases}$

This has many names:
* kernel
* retarded Green function
* retarded <u>propagator</u>

In general, $\Psi(\vec{r}'', t'') = \int d^3\vec{r}' \, K(\vec{r}'', t''; \vec{r}', t') \, \Psi(\vec{r}', t')$. (for $t'' > t'$)

Given the wavefunction at a fixed time $t'$, do an integral to find the wavefunction at any later time $t''$.

Suppose $H = -\frac{\hbar^2 \nabla^2}{2m} + V(\vec{r})$ (doesn't depend on $t$).

There are at least 3 ways to compute the propagator that look different, but are actually equivalent.

1) Suppose you know a complete set of $H$ eigenstates $|a\rangle$, with energies $E_a$. Then:

$$K(\vec{r}'', t''; \vec{r}', t') = \langle \vec{r}'', t'' | \vec{r}', t'\rangle \, \Theta(t'' - t')$$

Heaviside step function $= \begin{cases} 1 & \text{for } t'' > t' \\ 0 & \text{for } t'' < t' \end{cases}$

$$= \langle \vec{r}'', t' | e^{-iH(t''-t')/\hbar} | \vec{r}', t'\rangle \, \Theta(t'' - t')$$

same time, so can suppress

$$= \langle \vec{r}'' | e^{-iH(t''-t')/\hbar} | \vec{r}'\rangle \, \Theta(t'' - t')$$

$$= \sum_a \langle \vec{r}'' | \underbrace{e^{-iH(t''-t')/\hbar}}_{e^{-iE_a(t''-t')/\hbar}} | a\rangle \langle a | \vec{r}'\rangle \, \Theta(t'' - t')$$

$$= \sum_a e^{-iE_a(t''-t')/\hbar} \langle \vec{r}'' | a\rangle \langle a | \vec{r}'\rangle \, \Theta(t'' - t')$$

wavefunctions of energy eigenstates

2) $K$ satisfies a differential equation:

$$i\hbar \frac{\partial}{\partial t''} K(\vec r'', t''; \vec r', t') = i\hbar \frac{\partial}{\partial t''}\left[\langle \vec r'', t' | e^{-iH(t''-t')/\hbar} | \vec r', t'\rangle \, \Theta(t''-t')\right]$$

$$= \langle \vec r'', t'' | H | \vec r', t'\rangle \, \Theta(t''-t') + \underbrace{\langle \vec r'', t'' | \vec r', t'\rangle}_{\text{only matters when } t''=t'} \, \delta(t''-t') \, (i\hbar)$$

$\delta(t''-t')$ from $\frac{d}{dx}\Theta(x) = \delta(x)$

$$= \left(-\frac{\hbar^2 \nabla''^2}{2m} + V(\vec r'')\right) K(\vec r'', t''; \vec r', t') + \underbrace{\langle \vec r'' | \vec r'\rangle}_{\delta^{(3)}(\vec r - \vec r')} \, \delta(t''-t') \, (i\hbar)$$

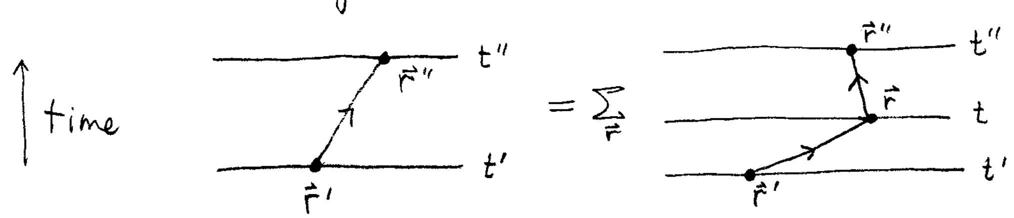
So: $$\left[-\frac{\hbar^2 \nabla''^2}{2m} + V(\vec r'') - i\hbar \frac{\partial}{\partial t''}\right] K(\vec r'', t''; \vec r', t') = -i\hbar \, \delta^{(3)}(\vec r - \vec r') \, \delta(t - t')$$

3) $$K(\vec r'', t'', \vec r', t') \sim \sum_{\substack{\text{paths} \\ \text{from } \vec r' \text{ at } t' \\ \text{to } \vec r'' \text{ at } t''}} (\text{something})$$

We'll make this more precise later. To motivate it, note:

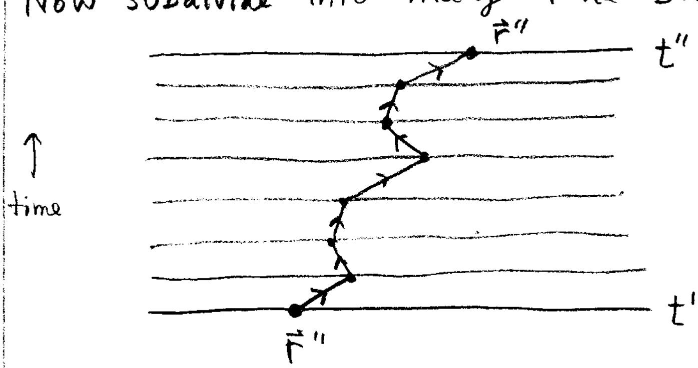
$$K(\vec r'', t''; \vec r', t') = \langle \vec r'', t'' | \vec r', t'\rangle = \int d^3\vec r \, \langle \vec r'', t'' | \vec r, t\rangle \langle \vec r, t | \vec r', t'\rangle$$

$$= \int d^3\vec r \; K(\vec r'', t''; \vec r, t) \, K(\vec r, t; \vec r', t')$$




To propagate from $t'$ to $t''$, add up all ways to propagate to $t$, then from $t$ to $t''$.

Now subdivide into many time steps:




or, as a formula ...

$$K(\vec{r}'',t'';\vec{r}',t') = \int d^3\vec{r}_1 \int d^3\vec{r}_2 \ldots \int d^3\vec{r}_n \; K(\vec{r}'',t'';\vec{r}_n,t_n)\, K(\vec{r}_n,t_n;\vec{r}_{n-1},t_{n-1})$$
$$\ldots K(\vec{r}_2,t_2;\vec{r}_1,t_1)\, K(\vec{r}_1,t_1;\vec{r},t)$$

This formula is not too useful in its present form, but motivates the idea that we sum over all possible ways to get from $\vec{r}'$ at $t'$ to $\vec{r}''$ at $t''$. Return to that idea shortly.

<u>Example</u>: Propagator for free particle in 1 dimension $(V(x)=0.)$

Use method #1. Energy eigenstates are momentum eigenstates.

$H|p\rangle = \frac{p^2}{2m}$. Continuous $p$ plays the role of $a$,

$$\sum_a \rightarrow \int_{-\infty}^{\infty} dp$$

So
$$K(x'',t'';x',t') = \int_{-\infty}^{\infty} dp \; e^{-iE_p(t''-t')/\hbar} \underbrace{\langle x''|p\rangle}_{\frac{e^{ipx''/\hbar}}{\sqrt{2\pi\hbar}}} \underbrace{\langle p|x'\rangle}_{\frac{e^{-ipx'/\hbar}}{\sqrt{2\pi\hbar}}} \Theta(t''-t')$$

$$= \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dp \; e^{-i(t''-t')p^2/2m\hbar} \, e^{ip(x''-x')/\hbar} \, \Theta(t''-t')$$

We need to compute $I(a,b) = \int_{-\infty}^{\infty} dp \; e^{-ap^2+bp}$,

then plug in $a = \frac{i(t''-t')}{2m\hbar}$ and $b = i(x''-x')/\hbar$.

To do the integral, complete the square in the exponent:

$$I(a,b) = \int_{-\infty}^{\infty} dp \; e^{-a(p+b/2a)^2} e^{b^2/2a}$$
$$= \underbrace{\left(\int_{-\infty}^{\infty} dp \; e^{-ap^2}\right)}_{\sqrt{\frac{\pi}{a}}} e^{b^2/2a}$$

So:
$$K(x'',t'';x',t') = \left[\frac{m}{2\pi i\hbar(t''-t')}\right]^{1/2} \exp\left[\frac{im(x''-x')^2}{2\hbar(t''-t')}\right] \Theta(t''-t')$$

for a free particle of mass $m$ in 1 dimension.

For a free particle moving in 3-d:

$$K(\vec{r}'',t''; \vec{r}',t') = \left[\frac{m}{2\pi i \hbar (t''-t')}\right]^{3/2} \exp\left[\frac{im|\vec{r}''-\vec{r}'|^2}{2\hbar(t''-t')}\right] \theta(t''-t')$$

<u>A harder example:</u> Harmonic oscillator in 1 dimension.

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2.$$ After some hard work:

$$K(x'',t'';x',t') = \left[\frac{m\omega}{2\pi i\hbar \sin(\omega(t''-t'))}\right]^{1/2} \exp\left[\frac{im\omega}{2\hbar\sin(\omega(t''-t'))}\left(\{x''^2+x'^2\}\cos(\omega(t''-t')) - 2x''x'\right)\right].$$

Notice this is periodic in $t''$, with $T = \frac{2\pi}{\omega}$.

Easiest way to get this is using method #1 again.
Now let's return to method #3, the Feynman path integral.

<u>Feynman's postulates of QM:</u>

A) $K(\vec{r}'',t''; \vec{r}',t')$ is the sum of an $\infty$ number of partial amplitudes, one for each space-time <u>path</u> connecting $\vec{r}'$ at $t'$ to $\vec{r}''$ at $t''$.

B) For a given path, specified by $\vec{r}(t)$ with $\vec{r}(t') = \vec{r}'$ and $\vec{r}(t'') = \vec{r}''$, the partial amplitude is:

$$\langle \vec{r}'',t''|\vec{r}',t'\rangle_{path} = X \exp\left[\frac{i}{\hbar} S[\vec{r}(t)]\right], \quad \text{where:}$$

$$S[\vec{r}(t)] = \begin{pmatrix}\text{classical action}\\ \text{for that path}\end{pmatrix} = \int_{t'}^{t''} \underbrace{L(\vec{r},\dot{\vec{r}},t)}_{\text{Classical Lagrangian.}}\, dt$$

So $$K(\vec{r}'',t''; \vec{r}',t') = \langle \vec{r}'',t''|\vec{r}',t'\rangle = X \sum_{\substack{\text{paths}\\ q(t)}} \exp\left[\frac{i}{\hbar}S[\vec{r}(t)]\right].$$

normalization to be found