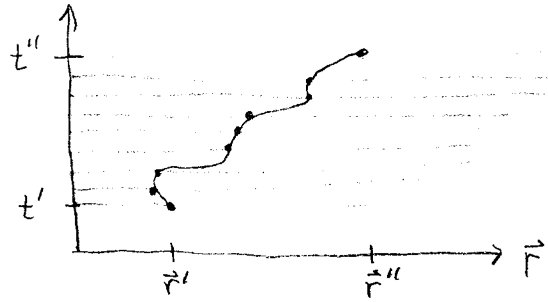
What is a "path"? A path is a function $\vec{r}(t)$ defined for $t' \leq t \leq t''$:



What is a sum over paths? Partition interval (t', t'') into $N-1$ equal steps of time ϵ. Then $t_j = t' + j\epsilon(t''-t')$ for $j = 0, 1, \ldots, N$. Now define values $\vec{r}_j$ at each t_j. Then:

$$S[\text{path}] = \sum_{j=1}^{N} \epsilon \left[\frac{m}{2} \left(\frac{|\vec{r}_j - \vec{r}_{j-1}|}{\epsilon} \right)^2 - V(\vec{r}_j) \right]$$

= action for trajectory with defined values connected by straight lines.

Now define:

$$\boxed{\langle \vec{r}'', t'' | \vec{r}', t' \rangle = \sum_{\text{paths}} e^{iS/\hbar} = \lim_{N\to\infty} \int \frac{d^3\vec{r}_1}{A} \int \frac{d^3\vec{r}_2}{A} \cdots \int \frac{d^3\vec{r}_{N-1}}{A} \, e^{iS(\vec{r}_1, \vec{r}_2, \ldots \vec{r}_N)/\hbar}}$$

A = normalization factor to be found.

Let's see if this is equivalent to Schrodinger's equation.

After one step forward $t'' = t' + \epsilon$ with ϵ small:

$$\Psi(\vec{r}, t+\epsilon) = \int \frac{d^3\vec{r}_1}{A} \, e^{iS(\vec{r}, \vec{r}_1)/\hbar} \, \Psi(\vec{r}_1, t) \qquad (\star)$$

Now expand both sides in small ϵ:

$$\Psi(\vec{r}, t+\epsilon) = \Psi(\vec{r}, t) + \epsilon \frac{\partial}{\partial t} \Psi(\vec{r}, t) + \frac{\epsilon^2}{2} \frac{\partial^2}{\partial t^2} \Psi(\vec{r}, t) + \ldots$$

For RHS: $S(\vec{r}, \vec{r}_1) = \epsilon \left(\frac{m}{2} \frac{|\vec{r} - \vec{r}_1|^2}{\epsilon^2} - V(\vec{r}) \right)$ = action for t' to $t' + \epsilon$.

So $$\text{RHS} = \int \frac{d^3\vec{r}_1}{A} \exp\left[\frac{i}{\hbar} \epsilon \left[\frac{m}{2} \left(\frac{|\vec{r} - \vec{r}_1|^2}{\epsilon^2} \right) - V(\vec{r}) \right] \right] \Psi(\vec{r}_1, t)$$

$$\text{RHS of } \maltese = \left[1-\frac{i\epsilon}{\hbar}V(\vec r)\right]\int\frac{d^3\vec r_1}{A}\exp\left[\frac{i}{\hbar}\left(\frac{m}{2\epsilon}\right)|\vec r-\vec r_1|^2\right]\psi(\vec r_1,t)$$

As $\epsilon\to 0$, there will be almost complete cancellation from the rapidly varying phase $e^{i(\text{stuff})/\epsilon}$. The only region that contributes as $\epsilon\to 0$ is where $(\text{stuff})\approx 0$, or $|\vec r-\vec r_1|^2 = \text{small}$.

Shift integration variable $\vec r_1\to\vec r_1+\vec r$:

$$\text{RHS of } \maltese = \left[1-\frac{i\epsilon}{\hbar}V(\vec r)\right]\int\frac{d^3\vec r_1}{A}\exp\left[\frac{i}{\hbar}\left(\frac{m}{2\epsilon}\right)r_1^2\right]\psi(\vec r+\vec r_1,t)$$

Since only small r_1 contributes, expand:

$$\psi(\vec r+\vec r_1,t) = \psi(\vec r,t)+\vec r_1\cdot\vec\nabla\psi(\vec r,t)+\frac{(\vec r_1\cdot\vec\nabla)^2}{2}\psi(\vec r,t)+\dots$$

$$\text{RHS of } \maltese = \left[1-\frac{i\epsilon}{\hbar}V(\vec r)\right]\int\frac{d^3\vec r_1}{A}\exp\left(\frac{i}{\hbar}\left(\frac{m}{2\epsilon}\right)r_1^2\right)\left[\psi(\vec r,t)+\underbrace{\vec r_1\cdot\vec\nabla\psi}_{\substack{\to 0,\\ \text{odd under}\\ \vec r_1\to-\vec r_1}}+\frac12\underbrace{(\vec r_1\cdot\vec\nabla)^2}_{\substack{\to\frac13\vec r_1^{\,2}\nabla^2\\ \text{(use rectangular}\\ \text{coordinates)}}}\psi+\dots\right]$$

So, putting together $\maltese$, expanded to order ϵ:

$$\psi(\vec r,t)+\epsilon\frac{\partial}{\partial t}\psi(\vec r,t) = \left[1-\frac{i\epsilon}{\hbar}V(\vec r)\right]\int\frac{d^3\vec r_1}{A}\exp\left(\frac{i}{\hbar}\left(\frac{m}{2\epsilon}\right)r_1^2\right)\left[\psi(\vec r,t)+\frac16 r_1^2\nabla^2\psi(\vec r,t)\right]$$

Now equate powers of ϵ:

$\boxed{\epsilon^0}$
$$\psi(\vec r,t) = \int\frac{d^3\vec r_1}{A}e^{\frac{im}{2\hbar\epsilon}r_1^2}\psi(\vec r,t)\quad\Rightarrow$$

$$A=\int d^3\vec r_1\, e^{imr_1^2/2\hbar\epsilon} = 4\pi\int_0^\infty r_1^2\,dr_1\, e^{imr_1^2/2\hbar\epsilon} = \pi^{3/2}\left(\frac{2i\hbar\epsilon}{m}\right)^{3/2}$$

or $\boxed{A=\left(\frac{2\pi i\hbar\epsilon}{m}\right)^{3/2}}$.

$\boxed{\epsilon^1}$
$$\frac{\partial}{\partial t}\psi = -\frac{i}{\hbar}V(\vec r)\psi-\underbrace{\int\frac{d^3\vec r_1}{A}e^{imr_1^2/2\hbar\epsilon}\frac{r_1^2}{2}}_{\frac{1}{\epsilon A}\frac{4\pi}{6}\int_0^\infty r_1^4 e^{imr_1^2/2\hbar\epsilon}dr_1 = \frac{i\hbar}{2m}}\nabla^2\psi$$

Multiply by $i\hbar$:

$$i\hbar \frac{\partial}{\partial t}\psi = -\frac{\hbar}{2m}\nabla^2\psi + V(\vec{r})\psi \quad = \text{Schrodinger's equation.}$$

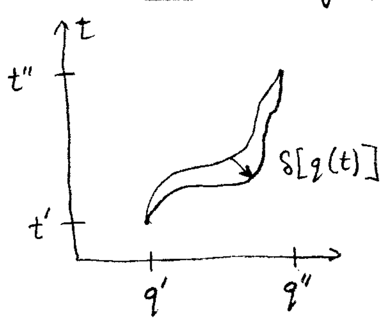
So Feynman's path integral $\Leftrightarrow$ Schrodinger QM. (Replace $\vec{r}$ by q = general coordinate(s).)

<u>Classical Limit</u> $\quad \langle q'', t'' | q', t' \rangle = \sum_{\text{paths}} e^{iS[q(t)]/\hbar}$

As $\hbar \to 0$, $S/\hbar$ is huge, rapidly varying $\Longrightarrow$ cancellation.

The main contribution is from <u>stationary paths</u> where $S[q(t)]$ is slowly varying.

<u>What is a stationary path?</u>



If we change $q(t) \to q(t) + \delta q(t)$, then $S \to S + \delta S$.

A <u>stationary path</u> has $\delta S = 0$ for <u>any</u> $\delta q(t)$.
<u>Dominates for $\hbar \to 0$.</u>

In QM, must sum over <u>all</u> paths.

In Classical, just go on ONE path, the stationary path $q(t)$ for which $\dfrac{\delta S[q(t)]}{\delta q(t)} = 0$ (this is called a functional derivative).

Only one path is followed for $\hbar \to 0$. Which one?

<u>Derive Classical Equations of motion from $\hbar \to 0$</u>

$$S[q(t)] = \int_{t'}^{t''} dt\, L(q, \dot{q}, t).$$

Make a change of path $q(t) \to q(t) + \delta q(t)$ with boundary conditions $\delta q(t') = \delta q(t'') = 0$. Then:

$$\delta S = \int_{t'}^{t''} dt \left[\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta\dot{q}\right]$$

Now $\delta\dot{q} = \frac{d}{dt}\delta q(t)$, so...

$$\delta S = \int_{t'}^{t''} dt \left[\frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \frac{d}{dt}(\delta q) \right].$$ Integrate last term by parts...

$$= \int_{t'}^{t''} dt \left[\frac{\partial L}{\partial q} \delta q - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) \delta q \right] + \underbrace{\left(\frac{\partial L}{\partial \dot{q}} \delta q \right)\Big|_{t'}^{t''}}_{=0}$$

$=0$ because of fixed endpoints $\delta q(t') = \delta q(t'') = 0$.

$$= \int_{t'}^{t''} dt \; \delta q \left[\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) \right].$$

In order to have $\delta S = 0$ for <u>all</u> δq, need

$$\boxed{\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0}$$ for paths that dominate as $\hbar \to 0$.

This is Lagrange's Equations of Motion for Classical mechanics. So we have derived classical mechanics from the $\hbar \to 0$ limit of Feynman's formulation of quantum mechanics.

More generally, if there are multiple coordinates q_i, then one can show as above that $$\boxed{\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0}$$ for each i.

<u>3-d Quantum Mechanics:</u> $q_1, q_2, q_3 \to \vec{r}$.

<u>Quantum Field Theory:</u> $q_i \to \phi(\vec{r})$ = a dynamical variable for each space point

Here $\vec{r}$ is a <u>label</u>, like i, not an operator.

But that's another story.