Tensor Operators are sets of operators that transform among each other in a special way under rotations.

Recall that vectors in rectangular coordinates obey:

$$V_i \longrightarrow \sum_{j=x,y,z} R_{ij} V_j$$

under a rotation parameterized by the orthogonal matrix R.

States transform like:

$$|\psi\rangle \rightarrow \mathcal{D}(R)\,|\psi\rangle$$

operator for rotation R.

So, the expectation value of a vector operator transforms like:

$$\langle\psi|V_i|\psi\rangle \rightarrow \langle\psi|\mathcal{D}^\dagger(R)\,V_i\,\mathcal{D}(R)|\psi\rangle$$

$$= \langle\psi|\sum_j R_{ij} V_j|\psi\rangle \qquad \text{(for \underline{any} state } |\psi\rangle)$$

So, $\mathcal{D}^\dagger(R)\,V_i\,\mathcal{D}(R) = \sum_j R_{ij} V_j$

For an infinitesimal rotation ϵ about an axis direction $\hat{n}$,

$\mathcal{D}(R) = 1 - \frac{i\epsilon}{\hbar}\vec{J}\cdot\hat{n}$. Plug this into previous equation:

$$V_i - \frac{i\epsilon}{\hbar}[V_i, \vec{J}\cdot\hat{n}] = \sum_j R_{ij} V_j$$

for ϵ about $\hat{n}$.

For example, if $\hat{n} = \hat{z}$, then $R = \begin{pmatrix} 1 & -\epsilon & 0 \\ +\epsilon & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

So, look at $i = x$: $\quad V_x - i\frac{\epsilon}{\hbar}[V_x, J_z] = V_x - \epsilon V_y.$

Therefore, $[V_x, J_z] = -i\hbar V_y.$

Looking at other combinations the same way gives

$$\boxed{[V_i, J_j] = i\hbar\,\epsilon_{ijk} V_k}$$ (summed over k)

Could take this as the defining requirement for V_i to be a vector.
(Can't just take any triplet of operators and call it a vector!)

For example, $V_i = J_i$ works. So does $V_i = R_i$, and $V_i = P_i$.

More generally, one can define <u>Cartesian tensors</u> as multi-index objects that satisfy:

$$T_{ijk\ldots} = R_{ii'} R_{jj'} R_{kk'} \ldots T_{i'j'k'\ldots} \qquad \text{(primed indices summed)}$$

The number of indices = <u>rank</u> of the tensor.

However, it is more convenient to discuss tensors in the spherical representation.

Recall how rotation operators act on angular momentum eigenstates:

$$\underset{\substack{\uparrow \\ \text{rotation} \\ \text{operator}}}{\mathcal{D}(R)} |\ell, m\rangle = \sum_{m'} |\ell, m'\rangle \underbrace{\mathcal{D}^{(\ell)}_{m'm}(R)}_{\substack{\text{numbers} \\ \text{(matrix elements)}}}$$

This motivates the following...

<u>Definition</u> A <u>spherical tensor operator of rank k</u> is a set of $2k+1$ operators $T^{(k)}_q$ $\quad (q = -k, -k+1, \ldots, k-1, k)$ that satisfy:

$$\mathcal{D}(R)\, T^{(k)}_q\, \mathcal{D}^\dagger(R) = \sum_{q'=-k}^{k} \mathcal{D}^{(k)}_{qq'}(R)\, T^{(k)}_{q'}$$

An alternate version is obtained by taking $\mathcal{D}(R) = 1 + \frac{i\epsilon}{\hbar} \vec{J} \cdot \hat{n}$ again. Plugging in, we get

$$[\vec{J} \cdot \hat{n},\, T^{(k)}_q] = \sum_{q'} T^{(k)}_{q'} \langle kq' | \vec{J} \cdot \hat{n} | kq \rangle$$

Or, using $\hat{n} = \hat{z}$ and then $\hat{n} = \hat{x} \pm i\hat{y}$,

$$[J_z,\, T^{(k)}_q] = \hbar q\, T^{(k)}_q$$

$$[J_\pm,\, T^{(k)}_q] = \hbar \sqrt{(k \mp q)(k \pm q + 1)}\; T^{(k)}_{q \pm 1}$$

Note the similarity between <u>operators</u> $T^{(k)}_q$ and <u>states</u> $|\underset{\substack{\uparrow \\ \ell}}{k}, \underset{\substack{\uparrow \\ m}}{q}\rangle$.

Example: Can form a dyadic rank 2 tensor from any two vectors V_i and U_i. Let $T_{ij} = U_i V_j$ = Cartesian rank 2. This is reducible; it can be divided into subsets which only transform among themselves under rotations:

$$T_{ij} = U_i V_j = \underbrace{\tfrac{1}{3} U\cdot V \delta_{ij}}_{\substack{\text{multiple of identity tensor} \\ (q=0)}} + \underbrace{\tfrac{1}{2}\left(U_i V_j - U_j V_i\right)}_{\substack{\text{antisymmetric tensor} \\ (q=1)}} + \underbrace{\left[\tfrac{1}{2}\left(U_i V_j + U_j V_i\right) - \tfrac{1}{3} U\cdot V \delta_{ij}\right]}_{\substack{\text{symmetric traceless} \\ (q=2)}}$$

Actually, these correspond to spherical tensors with $q = 0, 1, 2$.

In general, spherical tensors are nice because:

1) easier to work with in spherically-symmetric systems
2) irreducible

Writing $U_0 = U_z$, $U_{+1} = -\frac{(U_x + iU_y)}{\sqrt{2}}$, $U_{-1} = \frac{U_x - iU_y}{\sqrt{2}}$,

1 $\quad T_0^{(0)} = -\frac{1}{3} U\cdot V = \frac{1}{3}\left(-U_0 V_0 + U_{+1} V_{-1} + U_{-1} V_{+1}\right) \qquad (q=0)$

3 $\left\{ T_q^{(1)} = -\frac{i}{\sqrt{2}}\left(\vec{U}\times\vec{V}\right)_q \qquad (q = +1, 0, -1) \qquad \text{(note antisymmetric)}\right.$

$$5\left\{\begin{aligned} T_{\pm 2}^{(2)} &= U_{\pm 1} V_{\pm 1} \\ T_{\pm 1}^{(2)} &= \frac{1}{\sqrt{2}}\left(U_{\pm 1} V_0 + U_0 V_{\pm 1}\right) \\ T_0^{(2)} &= \frac{1}{\sqrt{6}}\left(2U_0 V_0 + U_{+1} V_{-1} + U_{-1} V_{+1}\right) \end{aligned}\right\} \text{ symmetric traceless}$$

(Not all rank 2 tensors can be formed by multiplying vectors, however.)

Given spherical tensors X, Z of ranks k_1 and k_2, can form a product tensor of rank k by:

$$T_q^{(k)} = \sum_{q_1}\sum_{q_2} \underbrace{\langle k_1 k_2 ; q_1 q_2 | k_1 k_2 ; k q\rangle}_{\text{Clebsch-Gordan coefficients like } \langle l_1 l_2 ; m_1 m_2 | l_1 l_2 ; l m\rangle} X_{q_1}^{(k_1)} Z_{q_2}^{(k_2)}$$

Proof: show that the given $T_q^{(k)}$ satisfies the defining transformation law:

$$\mathcal{D}^\dagger(R)\, T_q^{(k)}\, \mathcal{D}(R) = \sum_{q_1 q_2} \langle k_1 k_2 q_1 q_2 | k_1 k_2 k q \rangle\, \mathcal{D}^\dagger(R) X_{q_1}^{k_1} \mathcal{D}(R) \mathcal{D}^\dagger(R) Z_{q_2}^{k_2} \mathcal{D}(R)$$

$$= \sum_{q_1 q_2} \sum_{q_1' q_2'} \langle k_1 k_2 q_1 q_2 | k_1 k_2 k q \rangle\, X_{q_1'}^{k_1}\, \mathcal{D}_{q_1' q_1}^{(k_1)}(R^{-1})\, Z_{q_2'}^{k_2}\, \mathcal{D}_{q_2' q_2}^{(k_2)}(R^{-1})$$

Now recall the Clebsch-Gordon series: (Sakurai 3.7.69)

$$\mathcal{D}_{m_1 m_1'}^{(j_1)}\, \mathcal{D}_{m_2 m_2'}^{(j_2)} = \underbrace{\sum_{j}}_{\text{runs from } |j_1 - j_2| \text{ to } j_1 + j_2.} \sum_{m} \sum_{m'} \langle j_1 j_2 ; m_1 m_2 | j_1 j_2 ; j m \rangle \langle j_1 j_2 ; m_1' m_2' | j_1 j_2 ; j m' \rangle \mathcal{D}_{m m'}^{(j)}(R)$$

Use this with $(j_1, j_2, m_1, m_1', m_2, m_2', j, m, m') \to (k_1, k_2, q_1', q_1, q_2', q_2, k'', q', q'')$:

$$\mathcal{D}^\dagger(R)\, T_q^{(k)}\, \mathcal{D}(R) = \sum_{\substack{k'', q', q'' \\ q_1' q_2'}} \underbrace{\left(\sum_{q_1 q_2} \langle k_1 k_2 ; q_1 q_2 | k_1 k_2 ; k q \rangle \langle k_1 k_2 ; q_1 q_2 | k_1 k_2 ; k'' q'' \rangle \right)}_{= \delta_{kk''} \delta_{qq''} \text{ (C-G orthonormality; Sakurai 3.7.42)}} \langle k_1 k_2 ; q_1' q_2' | k_1 k_2 ; k'' q' \rangle \mathcal{D}_{q' q''}^{k''} X_{q_1'}^{k_1} X_{q_2'}^{k_2}$$

$$= \sum_{q'} \underbrace{\sum_{q_1 q_2} \langle k_1 k_2 ; q_1' q_2' | k_1 k_2 ; k q' \rangle\, X_{q_1'}^{(k_1)}\, X_{q_2'}^{(k_2)}}_{T_{q'}^{(k)}}\, \mathcal{D}_{q' q}^{(k)}(R^{-1})$$

$$= \sum_{q'} T_{q'}^{(k)}\, \mathcal{D}_{q' q}^{(k)}(R^{-1}) \quad = \text{defining transformation law under rotations } \checkmark$$

Note: same C-G coefficients appear as for addition of angular momentum.

Spherical tensors are "eigen-operators of angular momentum".

$$T_q^{(k)} \longleftrightarrow |\, l\, m \rangle$$

$$k \longleftrightarrow l$$

$$q \longleftrightarrow m$$

Selection rules for spherical tensor matrix elements.

Consider states $|\alpha, j, m\rangle$ that are eigenstates of H, J^2, J_z.

other e-values, for example n in the Hydrogen atom.

Then: $\langle \alpha', j', m' | T_q^{(k)} | \alpha, j, m \rangle = 0$ unless $|j-k| \leq j' \leq j+k$

<u>and</u> $m' = q+m$.

<u>Comments:</u> 1) These are necessary, not sufficient conditions for non-vanishing.

2) Note these are the same conditions for addition of angular momentum (j,m) and (k,q) to give (j',m').

More generally, the <u>Wigner–Eckhart Theorem</u> says:

$$\langle \alpha', j', m' | T_q^{(k)} | \alpha, j, m \rangle = \underbrace{\langle j k ; m q | j k ; j' m' \rangle}_{\text{Clebsch-Gordon coefficient}} \underbrace{\frac{1}{\sqrt{2j+1}}}_{\text{convention}} \underbrace{\langle \alpha' j' \| T^{(k)} \| \alpha j \rangle}_{\text{"reduced matrix element"}}$$

This <u>defines</u> the reduced matrix element, which is independent of m, m', and q. (Otherwise, the theorem would be content-free.)

So one can evaluate the reduced matrix element for some particularly convenient choice of m, m', q, and use that together with the C-G coefficients to get $\langle \alpha', j', m' | T_q^{(k)} | \alpha, j, m \rangle$ for <u>any</u> m, m', q.

The C-G coefficients are pure geometry (how are the states and operators oriented with respect to coordinates).

The reduced matrix element contains all the dynamics specific to the tensor operator $T^{(k)}$.

The selection rules now follow from those of the C-G coefficients.

Idea of proof: matrix elements of $T_q^{(k)}$ satisfy same recursion relations as C-G coefficients, so they must be proportional.

Proof: $\langle\alpha', j', m'|[J_\pm, T_q^{(k)}]|\alpha, j, m\rangle = \hbar\sqrt{(k\mp q)(k\pm q+1)}\ \langle\alpha', j', m'|T_{q\pm 1}^{(k)}|\alpha, j, m\rangle$

defining transformation law for tensor

$//$

$\sqrt{(j'\pm m')(j'\mp m'+1)}\ \langle\alpha', j', m'\mp 1|T_q^{(k)}|\alpha, j, m\rangle$

$-\sqrt{(j\mp m)(j\pm m+1)}\ \langle\alpha', j', m'|T_q^{(k)}|\alpha, j, m\pm 1\rangle$

This has the same form as recursion relations for $\langle jk; mq|jk; j'm'\rangle$. Both are first-order linear homogeneous equations with the same coefficients.

If $\sum_j a_{ij}x_j = 0$ and $\sum_j a_{ij}y_j = 0$, then $x_j = \underset{\substack{\uparrow\\ \text{independent of } j.}}{c}\, y_j$

So $\langle\alpha', j', m'|T_q^{(k)}|\alpha, j, m\rangle = \underbrace{\begin{pmatrix}\text{constant independent of}\\ q, m, m'\end{pmatrix}}_{\text{call this } \langle\alpha' j'\|T^{(k)}\|\alpha, j\rangle/\sqrt{2j+1}}\langle jk; mq|jk; j'm'\rangle$ ✓.

Example: Scalar operator (rank $k=0$) S:

$$\langle\alpha', j', m'|S|\alpha, j, m\rangle = \delta_{jj'}\,\delta_{mm'}\,\frac{1}{\sqrt{2j+1}}\langle\alpha', j\|S\|\alpha, j\rangle$$

evaluate once for $j=m=m'$, gives result for any m.

Example: Vector operator (rank $k=1$) V_0, V_{+1}, V_{-1}.

Then $\langle\alpha', j', m'|V_q|\alpha, j, m\rangle \neq 0$ requires:

$\Delta j = j'-j = 0$ or ± 1, and $\Delta m = m'-m = q = 0$ or ± 1, and j, j' not both 0. (The $0\to 0$ transition for j is forbidden).

This applies for electric dipole radiation, for example, with $V=R=$ position operator $R_0 = Z$, $R_{+1} = -\frac{1}{\sqrt{2}}(X+iY)$, $R_{-1} = \frac{1}{\sqrt{2}}(X-iY)$.

Also, for $j=j'\neq 0$, one can evaluate the reduced matrix element explicitly:

$$\langle\alpha', j, m'|V_q|\alpha, j, m\rangle = \frac{\langle\alpha', j, m|\vec{J}\cdot\vec{V}|\alpha, j, m\rangle}{\hbar^2 j(j+1)}\langle j, m'|J_q|j m\rangle$$

with $J_0 = J_z$ and $J_{\pm 1} = \mp\frac{1}{\sqrt{2}}(J_x \mp iJ_y) = \mp\frac{1}{\sqrt{2}}J_\pm$. (Proof in Sakurai, p. 241.)