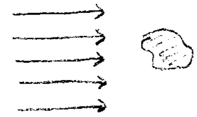
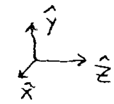
Differential Cross-Section

Consider particles impacting on a target.

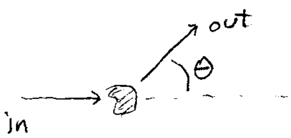




Suppose flux of incoming particles is:

$n \dfrac{\text{(particles)}}{\text{(area)(time)}}$ (assumed constant in time, and over area large in xy directions compared to target).

Count number of particles scattered in solid angle $d\Omega = d\phi \, d(\cos\theta)$



Let $s = \dfrac{\text{(particles)}}{\text{(time)}}$ between $\begin{cases} \phi \text{ and } \phi + d\phi \\ \cos\theta \text{ and } \cos\theta + d(\cos\theta) \end{cases}$

Define: $\dfrac{d\sigma}{d\Omega}$ = differential cross-section $= \dfrac{s}{n}\left(\dfrac{1}{d\Omega}\right)$

Note this has units of area.

Total cross-section: $\sigma = \displaystyle\int d\Omega \left(\frac{d\sigma}{d\Omega}\right) = \int_0^{2\pi} d\phi \int_{-1}^{1} d(\cos\theta) \, \frac{d\sigma}{d\Omega}$

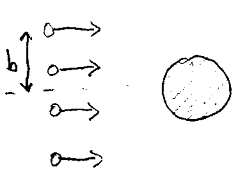
(also has units of area).

Very often, $\dfrac{d\sigma}{d\Omega}$ doesn't depend on ϕ. In that case:

$$\frac{d\sigma}{d(\cos\theta)} = \int_0^{2\pi} d\phi \, \frac{d\sigma}{d\Omega} = 2\pi \frac{d\sigma}{d\Omega}.$$

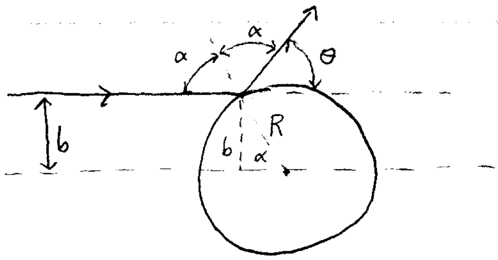
Intuitively, the cross-section = effective size of the target for that type of scattering.

<u>A classical physics example</u>: ping-pong balls (radius very small) scattering off a bowling ball (radius R, doesn't move when hit).



To find $\dfrac{d\sigma}{d\Omega} = \dfrac{1}{2\pi} \dfrac{d\sigma}{d(\cos\theta)}$, relate the impact parameter b to the scattering angle θ.

Assume impact parameter b is <u>random</u>, with a flat probability distribution.

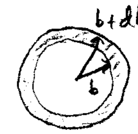


$$\alpha = \sin^{-1}\left(\frac{b}{R}\right)$$

$$\theta = \pi - 2\alpha = \pi - 2\sin^{-1}\left(\frac{b}{R}\right).$$

So $b = R\sin\left(\frac{\pi-\theta}{2}\right) = R\cos\frac{\theta}{2}$.

Let $n = \frac{(\text{particles})}{(\text{area})(\text{time})}$, then the number with impact parameter between b and $b+db$ is:



view towards target

$$\underbrace{n\,2\pi b\,db}_{\text{area of annulus}} = n\left(2\pi R\cos\left(\tfrac{\theta}{2}\right)\right)R\,d\left(\cos\left(\tfrac{\theta}{2}\right)\right)$$

$$= -\pi R^2 n\cos\left(\tfrac{\theta}{2}\right)\sin\left(\tfrac{\theta}{2}\right)d\theta = -\frac{\pi R^2 n}{2}\sin\theta\,d\theta$$

This is the number of $\frac{(\text{particles})}{(\text{time})}$ scattered between angle θ and $\theta + d\theta$.

So, using $-\sin\theta\,d\theta = d(\cos\theta)$,

$$s = \frac{(\text{particles})}{(\text{time})} \text{ between } \begin{cases}\phi \text{ and } \phi + d\phi \\ \cos\theta \text{ and } \cos\theta + d(\cos\theta)\end{cases} = \frac{\pi R^2 n}{2}\,d(\cos\theta)\,\frac{d\phi}{2\pi}$$

$$= \frac{R^2}{4}n\,d\Omega.$$

So $\frac{d\sigma}{d\Omega} = \frac{s}{n}\left(\frac{1}{d\Omega}\right) = \frac{\frac{R^2 n}{4}d\Omega}{n\,d\Omega} = \frac{R^2}{4} = \text{constant}.$

Also $\frac{d\sigma}{d(\cos\theta)} = 2\pi\frac{d\sigma}{d\Omega} = \frac{\pi R^2}{2}$, and

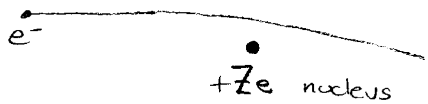
$$\sigma = \int_{-1}^{1}\frac{d\sigma}{d(\cos\theta)}\,d(\cos\theta) = \pi R^2.$$

This case has two unusual features:

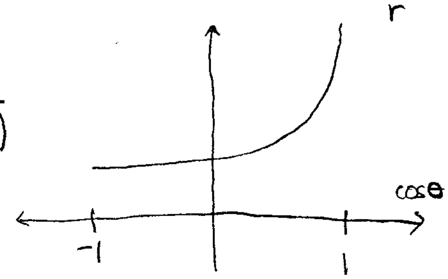
1) The differential cross-section is constant.

2) The total cross-section equals the geometrical cross-section (size of object).

Another example: Rutherford scattering. Coulomb potential $V(r) = \frac{Ze}{r}$.



$$\frac{d\sigma}{d\Omega} = \frac{Z^2 e^4}{16 E^2 \sin^4\left(\frac{\theta}{2}\right)}$$



Note $\frac{d\sigma}{d\Omega}$ blows up for small θ.

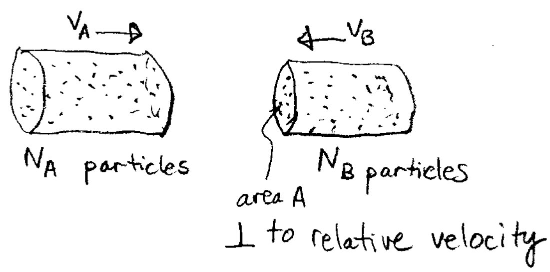
Only by defining a minimal scattering angle can we get a total cross-section:

$$\sigma(\theta > \theta_{min}) = \int_0^{2\pi} d\phi \int_{-1}^{\cos\theta_{min}} \frac{Z^2 e^4}{16 E^2 \sin^4\left(\frac{\theta}{2}\right)} \, d(\cos\theta)$$

$$= \frac{Z^2 e^4}{E^2} \left(\frac{\pi}{4}\right) \left(\frac{1 + \cos\theta_{min}}{1 - \cos\theta_{min}}\right)$$

This $\to \infty$ as $\theta_{min} \to 0$; the particle is <u>always</u> scattered at least a little bit.

Scattering of beams of particles



N_S = number of scattering events

$$\equiv \frac{N_A N_B}{A} \sigma$$

defines σ

The cross-section σ defined this way agrees with the above definition if one goes to the rest frame of beam B, takes them to be the target.

Units for σ = 1 barn $= 10^{-24}$ cm^2. 1 fb $= 10^{-15}$ b, 1 pb $= 10^{-12}$ b

Fermilab's Tevatron collides $p, \bar{p}$ at $E_{CM} = 1960$ GeV

Total data collected so far:

$$\frac{N_p N_{\bar{p}}}{A} \approx 6.1 \text{ fb}^{-1} = 6100 \text{ pb}^{-1} = \text{"integrated luminosity"}$$

At Tevatron $\sigma(p\bar{p} \to \text{anything}) \approx 0.075$ barns. ← fuzzy definition of "anything"

So N_s = total events $\approx (0.075 \text{ barns})(6.1 \text{ fb}^{-1})$

$$= (0.075)(6.1 \times 10^{15}) = 4.6 \times 10^{14} \text{ scattering events.}$$

Most are uninteresting, ignored.

For a more specific case, top-quark production:

$$\sigma(p\bar{p} \to t\bar{t}) = 6 \text{ pb}$$

So there should be about $(6 \text{ pb})(6100 \text{ pb}^{-1}) \approx 36{,}600$ $t\bar{t}$ events.

Unfortunately, most are lost.

Tevatron is looking for the Higgs boson h. If it exists, and $m_h = 130$ GeV, then:

$$\sigma(p\bar{p} \to hZ + \text{anything}) = 75 \text{ fb}.$$

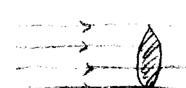
So there should be about $(75 \text{ fb})(6.1 \text{ fb}^{-1}) \approx 460$ Higgs events already. Unfortunately, most look too similar to other, boring, events.

Getting back to our solution for Lippmann-Schwinger:

$$\psi(\vec{r}) = \frac{1}{(2\pi)^{3/2}}\left[e^{i\vec{k}\cdot\vec{r}} + \frac{e^{ikr}}{r} f(\vec{k},\vec{k}')\right],$$

Recall $\left(\frac{\text{probability}}{\text{volume}}\right) = |\psi(\vec{r},t)|^2 = |e^{-iEt/\hbar}\,\psi(\vec{r})|^2 = |\psi(\vec{r})|^2$

So, in the initial beam,

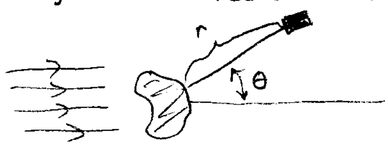


area $\perp$ to beam direction

$$\frac{(\text{total probability})}{(\text{area})} = \int_0^T \left|\frac{e^{i\vec{k}\cdot\vec{r}}}{(2\pi)^{3/2}}\right|^2 \underbrace{\underbrace{\left(\frac{\hbar k}{m}\right)}_{\text{velocity}} dt}_{\text{distance along beam}}$$

$$= \frac{1}{(2\pi)^3}\left(\frac{\hbar k}{m}\right)T$$

Now, a detector at (θ, ϕ) with solid angle $d\Omega$ will see:



Total probability of being scattered into $d\Omega$ =

$$\frac{1}{(2\pi)^{3/2}} \int_0^T \underbrace{\left| f(\vec{k}, \vec{k}') \frac{e^{ikr}}{r} \right|^2}_{\substack{\text{probability}\\ \text{volume}}} \underbrace{(r^2 \, d\Omega)}_{\text{area}} \underbrace{\underbrace{\frac{\hbar k}{m}}_{\text{velocity}} dt}_{\text{distance } \perp \text{ to area}} = \frac{|f(\vec{k}, \vec{k}')|^2}{(2\pi)^3} d\Omega \left(\frac{\hbar k}{m}\right) T$$

So
$$\frac{d\sigma}{d\Omega} = \frac{\frac{|f(\vec{k}, \vec{k}')|^2}{(2\pi)^3} d\Omega \left(\hbar k/m\right) T}{\frac{1}{(2\pi)^3} \left(\frac{\hbar k}{m}\right) T \, d\Omega} \quad , \text{ or}$$

$$\boxed{\frac{d\sigma}{d\Omega} = |f(\vec{k}, \vec{k}')|^2}$$

Note the scattering amplitude $f(\vec{k}, \vec{k}')$ has units of (length).

The Born Approximation

(Max Born = Olivia Newton-John's grandfather.)

Recall:

$$\psi(\vec{r}) = \frac{1}{(2\pi)^{3/2}} \left[e^{i\vec{k}\cdot\vec{r}} + \frac{e^{ikr}}{r} f(\vec{k}, \vec{k}') \right] \quad \text{where}$$

$$f(\vec{k}, \vec{k}') = -\frac{1}{4\pi} \left(\frac{2m}{\hbar^2}\right) (2\pi)^{3/2} \int d^3\vec{r}' \, e^{-i\vec{k}'\cdot\vec{r}'} V(\vec{r}') \, \psi(\vec{r}')$$

If the scattering inside the target is not too strong, then to first approximation, $\psi(\vec{r}') \approx \frac{1}{(2\pi)^{3/2}} e^{i\vec{k}\cdot\vec{r}'}$.

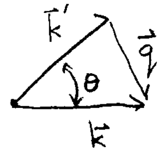
Plug this in:

$$f(\vec{k}, \vec{k}') = -\frac{1}{4\pi} \left(\frac{2m}{\hbar^2}\right) \underbrace{\int d^3\vec{r}' \, e^{i(\vec{k}-\vec{k}')\cdot\vec{r}'} V(\vec{r}')}_{\text{This is a 3-d Fourier transform of the target.}}$$

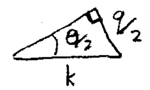
Recall $\vec{k} = k\hat{z}$ and $\vec{k}' = k\hat{r}$, where $E = \frac{\hbar^2 k^2}{2m}$.

An important special case: spherically symmetric potential.

Then $f(\vec{k},\vec{k}')$ only depends on $q \equiv |\vec{k}'-\vec{k}| = 2k\sin\left(\frac{\theta}{2}\right)$.



$\Rightarrow$



= right triangle.

Then f only depends on θ:

$$f(\theta) = f(\vec{k},\vec{k}') = -\frac{1}{4\pi}\left(\frac{2m}{\hbar^2}\right)\int_0^{2\pi} d\phi \int_{-1}^{1} d(\cos\theta)\int_0^\infty r^2 dr\, V(r)\, e^{iqr\cos\theta}$$

$$= -\frac{m}{\hbar^2}\int_0^\infty dr\, \frac{V(r)}{iqr}\left(e^{iqr}-e^{-iqr}\right)$$

$$= \frac{-2m}{\hbar^2}\frac{1}{q}\int_0^\infty r\,V(r)\sin(qr)\,dr$$

<u>Example</u>: Yukawa potential: $V(r) = X\dfrac{e^{-\mu r}}{r}$ (μ = range of potential, Sakurai writes $X = \frac{V_0}{\mu}$.)

Then $f(\theta) = \dfrac{-2m}{\hbar^2}\dfrac{1}{q}\displaystyle\int_0^\infty X e^{-\mu r}\sin(qr)\,dr = -\left(\frac{2mX}{\hbar^2}\right)\frac{1}{q^2+\mu^2}$

Also, use $q^2 = 4k^2\sin^2\left(\frac{\theta}{2}\right) = 2k^2(1-\cos\theta)$.

So $\dfrac{d\sigma}{d\Omega} = |f(\theta)|^2 = \left(\dfrac{2mX}{\hbar^2}\right)^2 \dfrac{1}{\left[2k^2(1-\cos\theta)+\mu^2\right]^2}$ in Born approximation.

Note that for $\begin{cases} X = Ze^2 \\ \mu = 0 \end{cases}$, we get back the Coulomb scattering problem.

Then $\dfrac{d\sigma}{d\Omega} = \dfrac{4m^2 Z^2 e^4}{\hbar^4}\left(\dfrac{1}{4k^2\sin^2\left(\frac{\theta}{2}\right)}\right)^2 = \dfrac{Z^2 e^4}{16E^2\sin^4\left(\theta/2\right)}$

using $E = \hbar^2 k^2/2m$.

This is the same as the <u>classical</u> Rutherford scattering result!

Comments on Born approximation for spherically symmetric potential:

1) $f(\theta)$ is always real
2) $f(\theta)$ is linear in $V(r)$, so $\frac{d\sigma}{d\Omega} = |f(\theta)|^2$ is independent of sign of V (for example, of charge in Rutherford scattering).
3) $f(\theta)$ (and therefore $\frac{d\sigma}{d\Omega}$) is only a function of $q = 2k^2(1-\cos\theta) = 4k^2 \sin(\frac{\theta}{2})$.
4) For low-energy limit ($k \to 0$, $q \to 0$),

$$f(\theta) \approx -\frac{1}{4\pi}\frac{2m}{\hbar^2}\int V(\vec{r})\, d^3\vec{r} = -\frac{2m}{\hbar^2}\int_0^\infty r^2 V(r)\, dr$$

does not depend on θ.

5) For large q (high energy limit with θ not too small), $f(\theta) \to 0$ because of oscillation of $\sin(qr)$ in integral.

When is the Born approximation valid?

Need $\Psi(\vec{r})$ not too different from $\frac{e^{i\vec{k}\cdot\vec{r}}}{(2\pi)^{3/2}}$, so correction term small. So need:

$$\left|\frac{1}{4\pi}\left(\frac{2m}{\hbar^2}\right)\int d^3\vec{r}\,' \frac{e^{i(\vec{k}'-\vec{k})\cdot\vec{r}}}{r} V(\vec{r}\,')\right| \ll 1.$$

This can be checked on a case-by-case basis.

A sufficient (but not always necessary) condition is just small enough $V(\vec{r}\,')$ everywhere.

For higher energies (large k), the Born approximation gets better because integrand oscillates more $\Rightarrow$ more concellation.

At low energies in the Yukawa potential case, $e^{i(\vec{k}'-\vec{k})\cdot\vec{r}} \to 1$, and $\frac{1}{r} \gtrsim \frac{1}{\mu}$ where $V(r)$ is non-zero, so the condition becomes $\frac{2m|X|}{\hbar^2\mu} \ll 1$. Compare to the condition necessary for a bound state to exist: $\frac{2m|X|}{\hbar^2\mu} \gtrsim 2.7$. If a bound state exists, Born fails at low E.