Born Approximation beyond leading order

Define the transition operator T by:

$T|\vec{k}\rangle = V|\psi^+\rangle$ for every plane wave ket $|\vec{k}\rangle$.

Now use Lippmann-Schwinger:

$$T|\vec{k}\rangle = V\left(|\vec{k}\rangle + \frac{1}{E-H_0+i\epsilon}\underbrace{V|\psi^+\rangle}_{T|\vec{k}\rangle}\right)$$

$$= \left(V + V\frac{1}{E-H_0+i\epsilon}T\right)|\vec{k}\rangle$$

Since true for every plane wave ket, we have an operator equation relating T, V, H_0:

$$T = V + V\frac{1}{E-H_0+i\epsilon}T.$$ Solve this for T:

$$\left(1 - V\frac{1}{E-H_0+i\epsilon}\right)T = V \quad \Rightarrow \quad T = \left(1 - V\frac{1}{E-H_0+i\epsilon}\right)^{-1} V$$

Now use $(1-X)^{-1} = 1 + X + X^2 + X^3 + \ldots$

$$T = V + V\frac{1}{E-H_0+i\epsilon}V + V\frac{1}{E-H_0+i\epsilon}V\frac{1}{E-H_0+i\epsilon}V + \ldots$$

Why is this useful?

Recall: $f(\vec{k},\vec{k}') = -\frac{1}{4\pi}(2\pi)^3\frac{2m}{\hbar}\langle\vec{k}'|\underbrace{V|\psi^+\rangle}_{T|\vec{k}\rangle}$ (Sakurai 7.1.34)

So $\boxed{f(\vec{k},\vec{k}') = -\frac{1}{4\pi}(2\pi)^3\frac{2m}{\hbar}\langle\vec{k}'|T|\vec{k}\rangle} = f_1(\vec{k},\vec{k}') + f_2(\vec{k},\vec{k}') + \ldots$

where $f_1(\vec{k},\vec{k}') = -\frac{1}{4\pi}(2\pi)^3\frac{2m}{\hbar}\langle\vec{k}'|V|\vec{k}\rangle$ (first-order Born)

$$f_2(\vec{k},\vec{k}') = -\frac{1}{4\pi}(2\pi)^3\frac{2m}{\hbar}\langle\vec{k}'|V\frac{1}{E-H_0+i\epsilon}V|\vec{k}\rangle$$

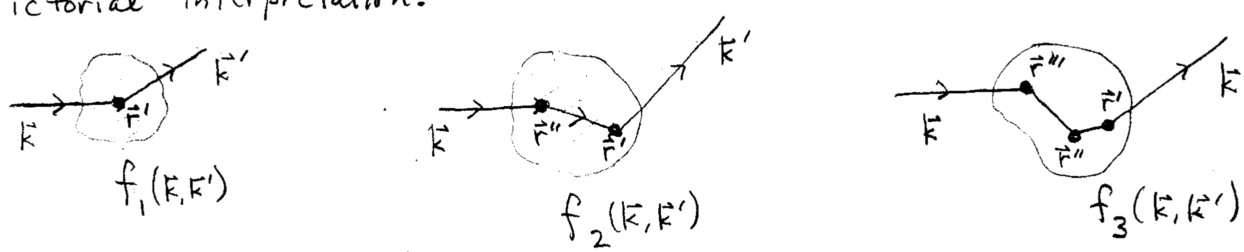
$$f_3(\vec{k},\vec{k}') = -\frac{1}{4\pi}(2\pi)^3\frac{2m}{\hbar}\langle\vec{k}'|V\frac{1}{E-H_0+i\epsilon}V\frac{1}{E-H_0+i\epsilon}V|\vec{k}\rangle$$ etc.

Explicitly: $$f_2(\vec k,\vec k') = -\frac{(2\pi)^3}{4\pi}\frac{2m}{\hbar^2}\int d^3\vec r'\int d^3\vec r''\ \underbrace{\langle \vec k'|\vec r'\rangle}_{\frac{1}{(2\pi)^{3/2}}e^{-i\vec k'\cdot\vec r'}}\, V(\vec r')\, \underbrace{\langle \vec r'|\frac{1}{E-H_0+i\epsilon}|\vec r''\rangle}_{\frac{2m}{\hbar^2}G_+(\vec r',\vec r'')}$$

$$V(\vec r'')\underbrace{\langle \vec r''|\vec k\rangle}_{\frac{1}{(2\pi)^{3/2}}e^{i\vec k\cdot\vec r''}}$$

So $$f_2(\vec k,\vec k') = \frac{-1}{4\pi}\left(\frac{2m}{\hbar^2}\right)^2\int d^3\vec r'\int d^3\vec r''\ e^{-i\vec k'\cdot\vec r'}\,V(\vec r')\,G_+(\vec r',\vec r'')\,V(\vec r'')\,e^{i\vec k\cdot\vec r''}$$

Pictorial interpretation:



$\bullet$ = interaction with V

$\rightarrow$ = propagation of particle $\begin{cases}\text{initial line:} & e^{i\vec k\cdot\vec r_i}\\ \text{final line:} & e^{-i\vec k\cdot\vec r_f}\\ \text{internal line:} & \left(\frac{2m}{\hbar^2}\right)G_+(\vec r_1,\vec r_2)\end{cases}$

<u>The Optical Theorem</u> relates the total cross-section and the forward scattering amplitude $f(\vec k,\vec k)\equiv f(\theta=0)$:

$$\boxed{\sigma_{tot} = \frac{4\pi}{k}\,\mathrm{Im}(f(0))}$$

Why should this be true? Conservation of probability.
The total scattered probability must come from depletion of the incident beam $\Rightarrow$ interference of initial beam amplitude and amplitude to scatter in the same direction $(\theta=0)$.

Usefulness: 1) provides a good check

2) Sometimes $f(0)$ is easier to calculate than $|f(\vec k,\vec k')|^2$.

surprisingly, not always!

Proof: (I'll follow Sakurai p. 391. A proof in Baym is trickier and requires tools we haven't introduced, but relates the theorem to the conservation of probability idea.)

Start with $f(0) = f(\vec{k},\vec{k}) = -\frac{1}{4\pi}\left(\frac{2m}{\hbar^2}\right)(2\pi)^3 \underbrace{\langle \vec{k}|T|\vec{k}\rangle}_{=\langle \vec{k}|V|\psi^+\rangle}$

Also use $\langle \vec{k}| = \langle \psi^+| - \langle \psi^+|V\frac{1}{E-H_0-i\epsilon}$. (Lippmann-Schwinger)

So

$$\mathrm{Im}(\langle \vec{k}|T|\vec{k}\rangle) = \underbrace{\mathrm{Im}(\langle\psi^+|V|\psi^+\rangle)}_{\substack{=0 \text{ because} \\ V \text{ is Hermitian}}} - \mathrm{Im}(\langle\psi^+|V\underbrace{\frac{1}{E-H_0+i\epsilon}}_{\text{not Hermitian}}V|\psi^+\rangle)$$

Now use $\frac{1}{x-i\epsilon} = \Pr\left(\frac{1}{x}\right) + i\pi\delta(x)$, where $\Pr\left(\frac{1}{x}\right) \equiv \begin{cases} 1/x & x\neq 0 \\ 0 & x=0\end{cases}$

(Principal value just ignores the singularity, which is captured by the δ-function.) So

$$\mathrm{Im}(\langle\vec{k}|T|\vec{k}\rangle) = -\mathrm{Im}(\langle\psi^+|V\underbrace{\Pr\left(\frac{1}{E-H_0}\right)}_{\text{Hermitian, so }\to 0}V|\psi^+\rangle) \underbrace{-\mathrm{Im}(\langle\psi^+|i\pi V\delta(E-H_0)V|\psi^+\rangle)}_{-\pi\langle\psi^+|V\delta(E-H_0)V|\psi^+\rangle}$$

$$= -\pi\langle\vec{k}|T^\dagger\delta(E-H_0)T|\vec{k}\rangle \quad \text{(used definition of } T \text{ again)}$$

$$= -\pi\underbrace{\int d^3k'}_{\substack{k'^2dk'd\Omega' \\ =\frac{1}{2}k'd(k'^2)d\Omega'}}\langle\vec{k}|T^\dagger|\vec{k}'\rangle\langle\vec{k}'|T|k\rangle\underbrace{\delta\left(E-\frac{\hbar^2k'^2}{2m}\right)}_{\delta\left(k'^2-\frac{2mE}{\hbar^2}\right)/\left(\frac{\hbar^2}{2m}\right)} \qquad \delta(f(x)) = \frac{\delta(x)}{|f'(0)|}$$

$$= -\pi\int d\Omega'\int_0^\infty d(k'^2)\ \frac{1}{2}\left(\frac{2m}{\hbar^2}\right)k'\,\delta(k'^2-k^2)\,|\langle\vec{k}'|T|\vec{k}\rangle|^2$$

$$= -\frac{\pi m}{\hbar^2}k\int d\Omega'\ \underbrace{|\langle\vec{k}'|T|\vec{k}\rangle|^2}_{\left|-4\pi\left(\frac{\hbar^2}{2m}\right)\frac{1}{(2\pi)^3}f(\vec{k},\vec{k}')\right|^2 = \frac{\hbar^4}{16\pi^4m^2}\frac{d\sigma}{d\Omega'}}$$

So $\text{Im}(f(0)) = -\frac{1}{4\pi}\left(\frac{2m}{\hbar^2}\right)(2\pi)^3 \underbrace{\text{Im}\left(\langle \vec{k} | T | \vec{k} \rangle\right)}_{-\frac{\pi m k}{\hbar^2}\left(\frac{\hbar^4}{16\pi^4 m^2}\right)\underbrace{\int d\Omega' \frac{d\sigma}{d\Omega'}}_{\sigma_{tot}}}$

$$= \frac{k}{4\pi}\sigma_{tot} \quad \checkmark$$

Spherical wave states for a free particle (Sakurai 7.5)

For a free particle of mass m, there are several useful bases for the ket space:

* Position eigenstates: $|\vec{r}\rangle$
* Momentum (plane-wave) eigenstates $|\vec{k}\rangle$ $\quad (\vec{k} = \vec{p}/\hbar)$
* Spherical wave states $|E, l, m\rangle$ (eigenstates of H_0, L^2, L_z)

The latter are convenient for scattering problems.

We'd like to be able to convert among these.

So, for example: $\quad |\vec{k}\rangle = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \int_0^\infty dE \; |E,l,m\rangle\langle E,l,m|\vec{k}\rangle$

and $\quad |E,l,m\rangle = \int d^3\vec{k} \; |\vec{k}\rangle\langle\vec{k}|E,l,m\rangle.$

So we want to know the matrix element:

$$\langle \vec{k} | E,l,m\rangle = \underbrace{g_{lE}(k)}_{\text{to be determined}} \underbrace{Y_{lm}(\theta,\phi)}$$

$\equiv Y_{lm}(\hat{k})$ for θ, ϕ corresponding to $\hat{k}$ direction

First, consider $\underbrace{\langle \vec{k} | (H_0 - E) | E,l,m\rangle}_{=0} = \langle \vec{k} | \left(\frac{\hbar^2 k^2}{2m} - E\right) | E,l,m\rangle$

from acting with H_0 to left

So $\langle \vec{k} | E,l,m\rangle \neq 0$ only if $\frac{\hbar^2 k^2}{2m} - E = 0.$

So $\quad g_{lE}(k) = N\,\delta\left(\frac{\hbar^2 k^2}{2m} - E\right) \qquad N = \text{normalization to be determined}$

To fix N, require: $\langle E', l', m' | E, l, m\rangle = \delta(E'-E)\,\delta_{ll'}\,\delta_{mm'}$.

Compute it by inserting a complete set of $|\vec{k}''\rangle$ states:

$$\langle E', l', m' | E, l, m\rangle = \int d^3k'' \,\langle E', l', m' | \vec{k}''\rangle\langle \vec{k}'' | E, l, m\rangle$$

$$= \int d\Omega_{k''} \int_0^\infty k''^2 dk'' \;|N|^2\, \delta\!\left(\frac{\hbar^2 k''^2}{2m} - E'\right) \delta\!\left(\frac{\hbar^2 k''^2}{2m} - E\right) Y^*_{l'm'}(\theta'', \phi'')\, Y_{lm}(\theta'', \phi'')$$

The angular integral is easy: $\int d\Omega_{k''}\, Y^*_{l'm'}(\theta'', \phi'')\, Y_{lm}(\theta'', \phi'') = \delta_{l'l}\,\delta_{m'm}$.

Now change variables: $E'' \equiv \dfrac{\hbar^2 k''^2}{2m}$, $\quad dE'' = \dfrac{\hbar^2}{m} k''\, dk''$.

So

$$\langle E', l', m' | E, l, m\rangle = |N|^2 \delta_{ll'}\delta_{mm'} \frac{m}{\hbar^2} \underbrace{\int_0^\infty dE''\, k''\, \delta(E''-E')\,\delta(E''-E)}_{k\,\delta(E'-E)}$$

$$= |N|^2 \frac{mk}{\hbar^2}\, \delta(E'-E)\,\delta_{ll'}\,\delta_{mm'}$$

So we need $N = \dfrac{\hbar}{\sqrt{mk}}$, and $\boxed{g_{lE}(k) = \dfrac{\hbar}{\sqrt{mk}}\, \delta\!\left(\dfrac{\hbar^2 k^2}{2m} - E\right) Y_{lm}(\hat{k})}$.

Therefore, $$\boxed{|\vec{k}\rangle = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \int_0^\infty dE\, \underbrace{\left(\frac{\hbar^3}{m\sqrt{2E}}\right)^{1/2}}_{\frac{\hbar}{\sqrt{mk}}} \underbrace{Y^*_{lm}(\theta, \phi)}_{\text{along } \hat{k} \text{ direction}}\, |E, l, m\rangle}$$

Note that all l, m contribute to any given plane-wave $|\vec{k}\rangle$.

So we've decomposed a general plane wave (for example, an initial state) into spherical waves.

Now we want to do the same for position eigenstates.

Start by considering the Schrodinger equation for a free particle in spherical coordinates:

$$\psi_{E,l,m}(r) = R_E(r)\, Y_{lm}(\theta, \phi).$$

Then $-\frac{\hbar^2}{2m}\nabla^2\psi = E\psi$ reduces to:

$$-\frac{\hbar^2}{2m}\frac{1}{r}\frac{d^2}{dr^2}(rR) + \frac{\hbar^2}{2m}\frac{\ell(\ell+1)}{r^2}R - ER = 0 \qquad \text{or}$$

$$\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{\ell(\ell+1)}{r^2} + k^2\right)R(r) = 0\,, \quad \text{where } k = \frac{\sqrt{2mE}}{\hbar}.$$

This is the spherical Bessel equation, with solutions:

$j_\ell(kr)$ and $n_\ell(kr)$, where

$$\left.\begin{aligned} j_\ell(x) &= \text{spherical Bessel function} \\ n_\ell(x) &= \text{spherical Neumann function} \end{aligned}\right\} \text{ for } \ell = 0, 1, 2, \ldots$$

The first few $j_\ell(x)$'s are:

$$j_0(x) = \frac{\sin(x)}{x} \qquad j_1(x) = \frac{\sin(x)}{x^2} - \frac{\cos(x)}{x} \qquad j_2(x) = \left(\frac{3}{x^3} - \frac{1}{x}\right)\sin(x) - \frac{3}{x^2}\cos(x)$$

These may appear to blow up as $x \to 0$ $(r \to 0)$, but actually:

$$j_\ell(x) \to \frac{x^\ell}{1\cdot 3\cdot 5\cdot \ldots (2\ell+1)} \quad \text{as } x \to 0.$$

Also,

$$n_0(x) = -\frac{\cos(x)}{x} \qquad n_1(x) = -\frac{\cos(x)}{x^2} - \frac{\sin(x)}{x} \qquad n_2(x) = \left(\frac{1}{x} - \frac{3}{x^2}\right)\cos(x) - \frac{3}{x^2}\sin(x)$$

These really do blow up at $x \to 0$, so we won't use them for a free particle.

In fact: $$\boxed{\langle \vec{r} \,|\, E, \ell, m\rangle = \frac{i^\ell}{\hbar}\sqrt{\frac{2mk}{\pi}}\, j_\ell(kr)\, Y_{\ell m}\underbrace{(\theta,\phi)}_{\text{for the } \hat{r} \text{ direction}}}$$

Where $k = \sqrt{2mE}/\hbar$.

Therefore,

$$|\vec{r}\rangle = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\int_0^\infty dE\ |E,\ell,m\rangle\langle E,\ell,m|\vec{r}\rangle$$

$$= \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\int_0^\infty dE\ \sqrt{\frac{2mk}{\pi}}\,\frac{(-i)^\ell}{\hbar}\, j_\ell(kr)\, Y^*_{\ell m}(\theta,\phi)\ |E,\ell,m\rangle$$

Partial Wave Expansion

Consider scattering from a spherical potential $V(r)$.

We will prove:

$$f(\theta) = f(\vec{k},\vec{k}') = \sum_{\underset{\text{angular momentum eigenvalue}}{l=0}}^{\infty} (2l+1)\, \underbrace{f_l(k)}_{\text{to be found}}\, \underbrace{P_l(\cos\theta)}_{\text{Legendre polynomials}}$$

$P_0(x)=1, \quad P_1(x)=x, \quad P_2(x)=\frac{3}{2}(x^2-1) \ldots$

Start with formula involving T ("transition") operator:

$$f(\vec{k},\vec{k}') = -\frac{1}{4\pi}\left(\frac{2m}{\hbar^2}\right)(2\pi)^3 \langle \vec{k}'|T|\vec{k}\rangle$$

$$= -\frac{1}{4\pi}\left(\frac{2m}{\hbar^2}\right)(2\pi)^3 \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\sum_{l'=0}^{\infty}\sum_{m'=-l'}^{l'} \int_0^\infty dE \int_0^\infty dE' \langle \vec{k}'|E',l',m'\rangle \underbrace{\langle E',l',m'|T|E,l,m\rangle}_{= T_l(E)\,\delta_{ll'}\,\delta_{mm'} \text{ by Wigner-Eckhart}} \langle E,l,m|\vec{k}\rangle$$

$$= -\frac{1}{4\pi}\left(\frac{2m}{\hbar^2}\right)(2\pi)^3 \sum_l \sum_m \int_0^\infty dE \int_0^\infty dE'\, T_l(E)\, \underbrace{\langle \vec{k}'|E',l,m\rangle}_{\frac{\hbar}{\sqrt{mk'}}\delta\left(E-\frac{\hbar^2k'^2}{2m}\right)Y_{lm}(\hat{k}')} \underbrace{\langle E,l,m|\vec{k}\rangle}_{\frac{\hbar}{\sqrt{mk}}\delta\left(E-\frac{\hbar^2k^2}{2m}\right)Y_{lm}(\hat{k})}$$

$$= -\frac{1}{4\pi}\left(\frac{2m}{\hbar^2}\right)(2\pi)^3\left(\frac{\hbar^2}{mk}\right) T_l(E) \sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y_{lm}(\hat{k}')\, Y_{lm}(\hat{k})$$

Take $\hat{k}$ along z-direction, then $Y_{lm}(\hat{k}) = Y_{lm}(\theta=0,\phi=0) = \sqrt{\frac{2l+1}{4\pi}}\,\delta_{m,0}$.

So $$\sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y_{lm}(\hat{k}')Y_{lm}(\hat{k}) = \sum_{l=0}^{\infty}\sqrt{\frac{2l+1}{4\pi}}\, \underbrace{Y_{l0}(\hat{k}')}_{\sqrt{\frac{2l+1}{4\pi}}P_l(\cos\theta)}$$

angle between $\vec{k}$ and $\vec{k}'$.

$$= \sum_{l=0}^{\infty} \frac{2l+1}{4\pi} P_l(\cos\theta)$$

Therefore: $$f(\vec{k},\vec{k}') = \sum_{l=0}^{\infty}(2l+1)\left(\frac{-\pi T_l(E)}{k}\right) P_l(\cos\theta)$$

So define $f_l(k) = -\frac{\pi T_l(E)}{k}$. ✓

To better understand the partial wave expansion, recall:

$$e^{ikz} = \sum_{l=0}^{\infty} (2l+1)\, \underbrace{i^l j_l(kr)}_{\frac{e^{ikr} - e^{-ikr+il\pi}}{2ikr}} P_l(\cos\theta) \quad \text{for large } r$$

Plug this into Lippmann-Schwinger:

$$\psi(\vec{r}) = \frac{1}{(2\pi)^{3/2}}\left[e^{ikz} + f(\theta)\frac{e^{ikr}}{r}\right] \quad \text{at large } r:$$

$$= \frac{1}{(2\pi)^{3/2}}\left[\sum_{l=0}^{\infty}(2l+1)P_l(\cos\theta)\left(\frac{e^{ikr} - e^{-ikr+il\pi}}{2ikr}\right) + \sum_{l=0}^{\infty}(2l+1)P_l(\cos\theta)\, f_l(k)\,\frac{e^{ikr}}{r}\right]$$

This is begging us to combine terms:

$$\psi(\vec{r}) = \frac{1}{(2\pi)^{3/2}}\sum_{l=0}^{\infty}(2l+1)\frac{P_l(\cos\theta)}{2ik}\left(\underbrace{[1+2ik f_l(k)]\frac{e^{ikr}}{r}}_{\text{outgoing spherical wave}} - \underbrace{\frac{e^{-ikr+il\pi}}{r}}_{\text{incoming spherical wave}}\right)$$

Note that the outgoing wave depends on $V(r)$ through $f_l(k)$, but the incoming wave doesn't depend on the scattering.

A crucial fact: $[1+2ik f_l(k)]$ is a pure phase $= e^{2i\delta_l}$.

(The 2 is a historical convention.)

Proof: Probability is conserved (potential just scatters, no absorption)

So the probability current density (Sakurai p.101)

$$\vec{j}(\vec{r},t) = \frac{\hbar}{m}\,\mathrm{Im}(\psi^* \vec{\nabla}\psi) \quad \text{must be conserved:}$$

$$\vec{\nabla}\cdot\vec{j} = 0$$

Use Gauss' Law on a sphere at large r: $\displaystyle\oint_{\substack{\text{large } r\\ \text{sphere}}} \vec{j}\cdot d\vec{a} = 0$

Applying this to any $\Psi(\vec{r}) = \sum_{\ell=0}^{\infty} P_\ell(\cos\theta)\left(a_\ell \frac{e^{ikr}}{r} + b_\ell \frac{e^{-ikr}}{r}\right)$,

one finds that $\oint \vec{j}\cdot d\vec{a} = \int j_r \, da = \frac{\hbar}{m}\int \mathrm{Im}\left(\Psi^* \frac{\partial}{\partial r}\Psi\right) r^2 d\Omega$

vanishes if and only if $|a_\ell| = |b_\ell|$ for <u>each</u> ℓ. ✓

So define:

$$1 + 2ik\, f_\ell(k) = \underbrace{S_\ell(k)}_{} = e^{2i\delta_\ell} \qquad \delta_\ell = \text{phase shift of the } \ell\text{'th partial wave}$$

matrix elements of a <u>unitary</u> operator S:

$$S_\ell(k) = \langle E,\ell,m|S|E,\ell,m\rangle$$

The scattering problem reduces to finding the phase shifts δ_ℓ (often only for the lowest few $\ell = 0, 1, 2, \dots$).

Other ways to write this:

$$f_\ell(k) = \frac{e^{2i\delta_\ell} - 1}{2ik} = \frac{e^{i\delta_\ell}\sin(\delta_\ell)}{k} = \frac{1}{k\cot(\delta_\ell) - ik}.$$

In any case:

$$f(\theta) = f(\vec{k}, \vec{k}') = \frac{1}{k}\sum_{\ell=0}^{\infty}(2\ell+1)\, e^{i\delta_\ell}\sin(\delta_\ell)\, P_\ell(\cos\theta)$$

Now let's compute σ_{tot} in terms of the δ_ℓ.

$$\sigma_{tot} = \int d\Omega\, |f(\theta)|^2 = \frac{1}{k^2}\sum_\ell \sum_{\ell'} (2\ell+1)(2\ell'+1)\, e^{i\delta_\ell}\sin(\delta_\ell)\, e^{-i\delta_{\ell'}}\sin(\delta_{\ell'})$$

$$\underbrace{\int_0^{2\pi} d\phi}_{2\pi}\; \underbrace{\int_{-1}^{1} d(\cos\theta)\, P_{\ell'}(\cos\theta)\, P_\ell(\cos\theta)}_{\frac{2}{2\ell+1}\delta_{\ell,\ell'}}$$

$$\boxed{\sigma_{tot} = \frac{4\pi}{k^2}\sum_{\ell=0}^{\infty}(2\ell+1)\sin^2(\delta_\ell)}$$

Check this with the Optical Theorem:

$$\sigma_{tot} = \frac{4\pi}{k}\,\mathrm{Im}(f(\theta=0)) = \frac{4\pi}{k}\left(\frac{1}{k}\sum_{\ell=0}^{\infty}(2\ell+1)\sin^2(\delta_\ell)\,\underbrace{P_\ell(1)}_{1}\right) = \frac{4\pi}{k^2}\sum_{\ell=0}^{\infty}(2\ell+1)\sin^2(\delta_\ell).$$

An important consequence is <u>partial wave unitarity</u>.

Write $\sigma_{tot} = \sum_{l=0}^{\infty} \sigma_l$ ← contribution from angular momentum l.

Then for each partial wave l:

$$\sigma_l = \frac{4\pi}{k^2}(2l+1)\sin^2(\delta_l) \leqslant \frac{4\pi}{k^2}(2l+1).$$

For any given l, the contribution to σ is bounded, regardless of the potential that determines δ_l.

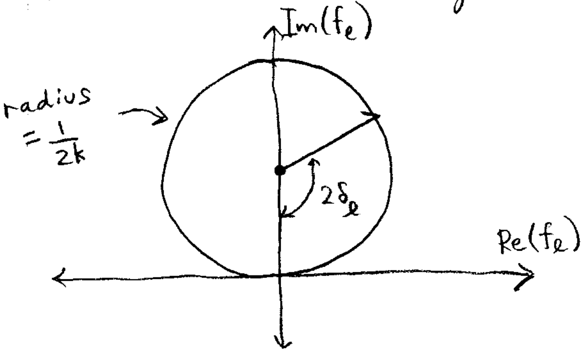
The bound decreases with energy:

$$\sigma_l < \frac{2\pi\hbar^2(2l+1)}{mE}.$$

This relies only on conservation of probability.

(One argument for the existence of the Higgs boson is that without it, $WW \to WW$ scattering doesn't obey partial wave unitarity. But there are other possibilities that fix the problem by adding new particles.)

Partial wave unitarity restricts what values $f_l(k)$ can take:



← is the unitary circle for $f_l(k)$

For <u>small</u> δ_l: $f_l(k)$ is near bottom, almost purely real.
(Recall Born approximation.)

For <u>maximal</u> $|f_l(k)| \simeq \frac{1}{k}$, $f_l(k)$ is almost pure imaginary (top of circle)

This can occur if the l'th partial wave is near a resonance. Also, $\sin(\delta_l) \approx 1$, σ_l saturates the partial wave unitarity bound.